On the Origin of Symmetry

Symplectic geometry & Noether's theorem

Aditya Dwarkesh

DMS Day 2024

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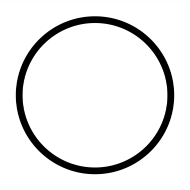
The Concept of a Manifold

Locally Euclidean



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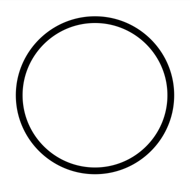

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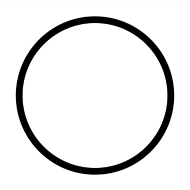
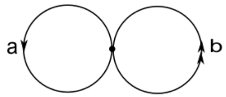
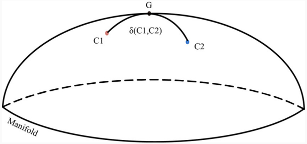




Structures on Manifolds

Lengths and Areas



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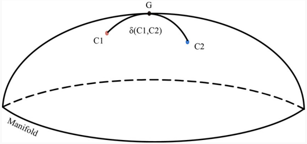

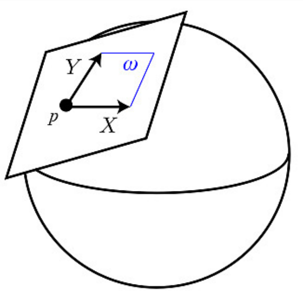

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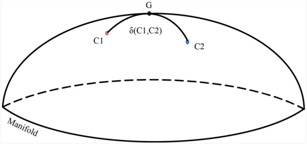
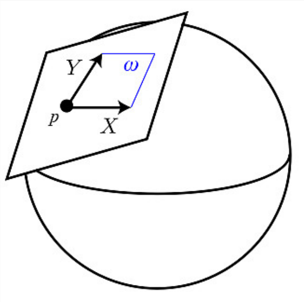
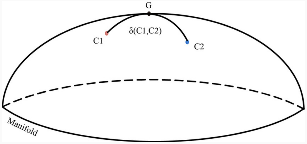
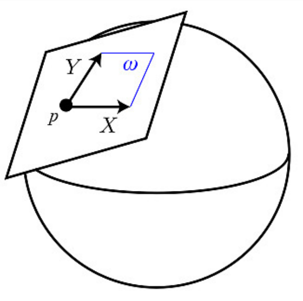




On $\mathbb{R}^2 : \omega = dxdy$

Lengths and Areas





On $\mathbb{R}^2 : \omega = dxdy$

On $S^2 : \omega = dhd\theta$

Classical mechanics on symplectic manifolds

Phase space: 3 position coordinates + 3 momentum coordinates + dynamical laws

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Generalization from $(\mathbb{R}^6)^n$: A smooth manifold M, with some additional structure encoding dynamics.

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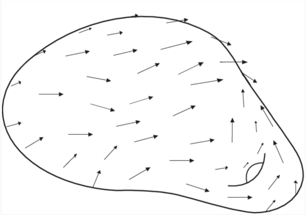
Generalization from $(\mathbb{R}^6)^n$: A smooth manifold M, with some additional structure encoding dynamics.

Question: How does a symplectic form produce dynamical laws on the manifold?

Dynamics

Vector fields



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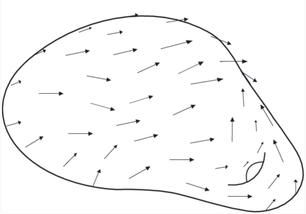
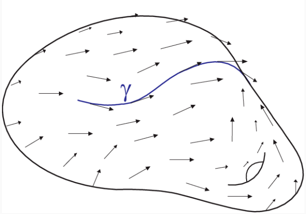
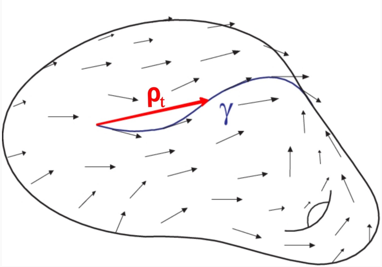




Flow



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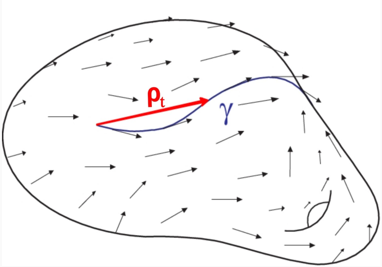

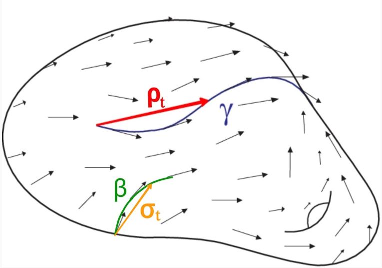

Hamiltonian vector field

Through the following equation, a given function $f : M \to \mathbb{R}$ is associated to a vector field X_f :

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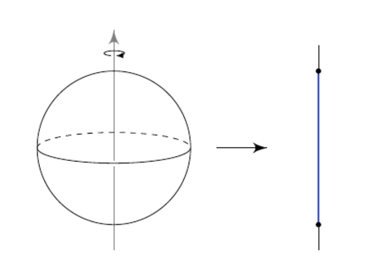
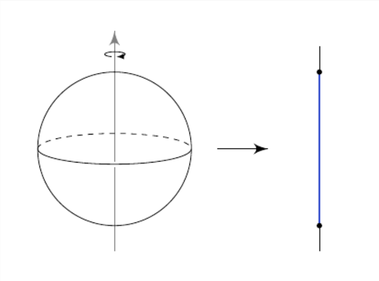
$$\boxed{i_{X_f}(\omega) + df = 0}$$

Hamiltonian vector field: Example



Height function.

Hamiltonian vector field: Example



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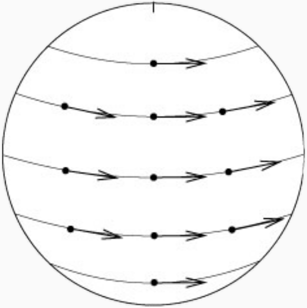
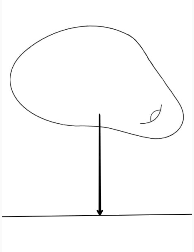


Hamiltonian vector field of the height function.

Mind map



Function.

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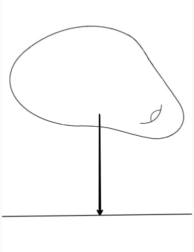

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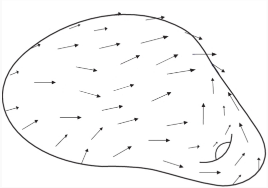

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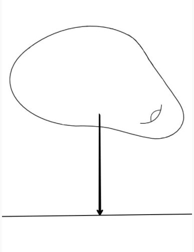

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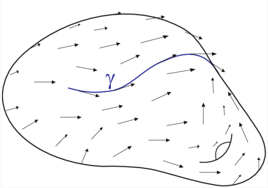
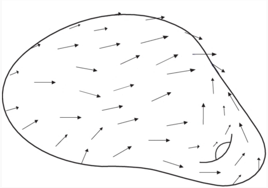


Integral curve.



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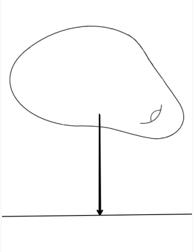

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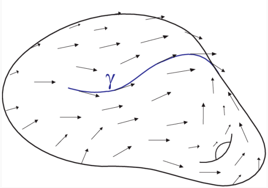

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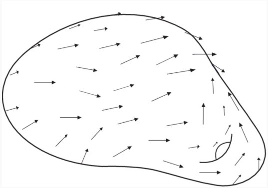

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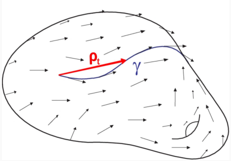

Flow.

Dynamics recovered

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Answer: Given a Hamiltonian function f, the symplectic form ω associates to it a vector field X_f, whose flow describes the time-evolution of the system.

Noether's Theorem

Conserved quantities

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- $f \circ \gamma_H = C$ (Constant on integral curves)

Symmetries

- (M, ω, H): Hamiltonian system
- V: Vector field with flow ρ_t

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$$\boxed{H = H \circ \rho_t}$$

Noether's theorem

Theorem: Let (M, ω, H) be a Hamiltonian system. If f is a conserved quantity, its Hamiltonian vector field is a symmetry.

Emmy Noether, 1882–1935.

Thank you!