Analysis-III

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Spivak's Calculus on Manifolds, Chapter II

2-1. Let $lim_{h\to 0}\frac{|f(a+h)-f(a)|}{|h|} = |Df(a)| \implies lim_{h\to 0}|f(a+h)-f(a)| = lim_{h\to 0}|hDf(a)| \leq lim_{h\to 0}|hM|$(for some $M \in \mathbb{R}$) $\implies lim_{h\to 0}|f(a+h)-f(a)| = 0$. Hence, proved.

2-2. If f is independent of the second variable, we can define $g(x) = f(x,0)$. Conversely, if $f(x,y_1) = g(x) = f(x,y_2)$, f is independent of the second variable.
$f'(a,b) = (g'(a),0)$.

2-3. f is independent of the first variable if $f'(a,b) = (0,g'(b))$, where $g:\mathbb{R}\to\mathbb{R}$. If f is independent of both variables, it must be the constant function.

2-4.

1. $h(t) = f(tx) = |tx|g(\frac{tx}{||tx||}) = tf(x)$, since g is an odd function. Thus, $h'(t) = f(x)$.
2. $lim_{h\to 0}\frac{|f(h,0)-f(0,0)|}{|(h,0)|} = lim_{h\to 0}g(\frac{(h,0)}{|h|}) = g(\pm 1,0) = 0$ (given). Similarly, $g(0,\pm 1) = 0 \implies lim_{k\to 0}\frac{|f(0,k)-f(0,0)|}{|(0,k)|} = 0$. Thus, if it exists, $Df(0,0) = 0$.
   However, $lim_{(h,k)\to 0}\frac{|(f(h,k)-f(0,0)|}{|(h,k)|} = lim_{x\to 0}g(\frac{x}{||x||})$ (replacing (h,k) with x). Since g is an odd function, this limit exists only if $g=0$. Hence, proved.

2-5. Defining $g(\frac{(x,y)}{|(x,y)|}) = \frac{x|y|}{|(x,y)|^2}$ shows the required.

2-6. It is clear that $lim_{x\to 0}\frac{|f(x,0)|}{|x|} = lim_{y\to 0}\frac{|f(0,y)|}{|y|} = 0$. Thus, if it exists, the derivative is 0.
However, $lim_{h\to 0}\frac{|f(h,h)|}{|(h,h)|} = \frac{1}{\sqrt{2}} \neq 0$. Thus, the derivative does not exist.

2-7. Note that $f(0) = 0$. Thus, $lim_{h\to 0}\frac{|f(h)-f(0)|}{|h|} \leq lim_{h\to 0}|h| = 0$. Thus, $Df(0) = 0$.

2-8. First, suppose f is differentiable. Note that $f_j = p_j o f$, where p is the projection map onto the first component. Being a linear map, p_j is differentiable. Thus, being the composition of two differentiable functions, f_j is differentiable.
Conversely, suppose each f_j is differentiable. Then, for $\lambda = (Df_1(a),...Df_n(a)), lim_{h\to 0}\frac{||f(a+h)-f(a)-\lambda h||}{||h||} \leq \sum_{j=1}^{n} lim_{h\to 0}\frac{|f_j(a+h)-f_j(a)-Df_j(a)|}{||h||} = 0$ (using problem 1-1 and interchanging limit and finite sum).

2-9.

1. If f is differentiable, we can define the function as $g(x) = f(a) + f'(a)(x-a)$.
   Alternatively, we have $lim_{h\to 0}\frac{f(a+h)-a_0}{h} = a_1$; but also, $f(a) = g(a) = a_0$. Thus, this gives $f'(a) = a_1$, and we are done.
2. By Taylor's theorem, $f(x) = \frac{f^n(y)}{n!}(x-a)^n + \sum_{i=1}^{n-1}\frac{f^i(x)}{i!}(x-a)^i, y \in (x,a)$. Thus, $\lim_{x\to a}\frac{f(x)-g(x)}{(x-a)^n} = lim_{x\to a}\frac{f^n(y)(x-a)^n - f^n(x)(x-a)^n}{(x-a)^n} = 0$.

2-10.

1. $\begin{pmatrix} yx^{y-1} & x^y ln(x) & 0 \end{pmatrix}$
2. $\begin{pmatrix} yx^{y-1} & x^y lnx & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. $\begin{pmatrix} sin(y)cos(xsin(y)) & xcos(y)cos(xsin(y)) \end{pmatrix}$
4. $cos(x(sin(ysin(z))) \begin{pmatrix} (sin(ysin(z)) & xcos(ysin(z))sin(z) & xycos(ysin(z))cos(z) \end{pmatrix}$
5. $\begin{pmatrix} y^z x^{y^z-1} & zy^{z-1}x^y ln(x) & y(x^{y^z})ln(x) \end{pmatrix}$
6. $\begin{pmatrix} (y+z)x^{y+z-1} & x^{y+z}ln(x) & x^{y+z}ln(x) \end{pmatrix}$
7. $\begin{pmatrix} z(x+y)^{z-1} & z(x+y)^{z-1} & (x+y)^z ln(x+y) \end{pmatrix}$
8. $\begin{pmatrix} ycos(xy) & xcos(xy) \end{pmatrix}$
9. $\begin{pmatrix} cos(3)xcos(xy)sin^{cos(3)-1}(xy) & cos(3)ycos(xy)sin^{cos(3)-1}(xy) \end{pmatrix}$
10. By 1, 3 and 8, we have: $\begin{pmatrix} ycos(xy) & xcos(xy) \\ sin(y)cos(xsin(y)) & xcos(y)cos(xsin(y)) \\ yx^{y-1} & x^y ln(x) \end{pmatrix}$

2-11. Let $h(t) = \int_a^t g$. We know that $h'(x) = g(x)$.

1. $f(x,y) = h(x+y) \implies Df(x,y) = (g(x+y), g(x+y))$.
2. $f(x,y) = h(xy) \implies Df(x,y) = (yg(xy), xg(xy))$.
3. $f(x,y,z) = h(sin(xsin(ysinz))) = h(r) \implies Df(x,y) = (a_1 g(r), a_2 g(r), a_3 g(r))$, where a_1, a_2, a_3 are the components of the matrix in 2-10 4.

2-12.

1. We shall first show that $|f(h,k)| \leq M|h||k|$, where $M = \sum_{i,j} |f(e_i, e_j)|$, where $\{e_i\}$ are the basis vectors for $\mathbb{R}^n$ and $\{e_j\}$ for $\mathbb{R}^m$.
   Let $h = \sum_i a_i e_i, k = \sum_j b_j e_j$. Then, $|f(h,k)| = |f(\sum_i a_i e_i, \sum_j b_j e_j)| = |\sum_{i,j} a_i b_j f(e_i, e_j)| \leq |\max\{a_i\}\max\{b_j\}|M \leq M|h||k|$.
   Now, note that $\frac{|h||k|}{|(h,k)|} = \frac{|h||k|}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2}$, since $(|h|-|k|)^2 \geq 0 \implies |h|^2 + |k|^2 \geq |h||k| \implies |h||k| \leq h^2 + k^2$.
   Thus, $0 \leq lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} \leq lim_{(h,k)\to 0} \frac{M|h||k|}{|(h,k)|} \leq lim_{(h,k)\to 0} M\sqrt{h^2+k^2} = 0$. Hence, proved.
2. Let $\lambda(h,k) = f(a,k) + f(h,b)$. Note that this is linear: $\lambda(c(h,k)+(x,y)) = f(a, ck+y) + f(ch+x, b) = cf(a,k) + f(a,y) + cf(h,b) + f(x,b) = c\lambda(h,k) + \lambda(x,y)$. Then, $lim_{(h,k)\to 0} \frac{f(a+h,b+k)-f(a,b)-\lambda(h,k)}{|(h,k)|} = lim_{(h,k)\to 0} \frac{|f(a,b)+f(a,k)+f(h,b)+f(h,k)-f(a,b)-\lambda(h,k)|}{|(h,k)|} = lim_{(h,k)\to 0} \frac{f(a,k)+f(h,b)-\lambda(h,k)}{|(h,k)|} = 0$.
   Thus, $Df(a,b)(x,y) = f(a,y) + f(x,b)$.
3. It is easy to check that $p(a,b) = ab$ is bilinear: $p(ca,b) = cab = cp(a,b) = p(a,cb); p(a+c,b) = ab + cb = p(a,b) + cb; p(a, b+d) = ab + ad = p(a,b) + p(a,d)$. Setting $f(h,k) = hk$, we obtain $Df(a,b)(x,y) = f(a,y) + f(x,b) = ay + bx$.

2-13.

1. From the above, we have $D(IP)(a,b)(x,y) = \langle a, y\rangle + \langle x, b\rangle$.
   $(IP)'(a,b) = (b,a)$, since $(b,a)(x,y)^t = \langle b, x\rangle + \langle a, y\rangle$.

2. Let $h(t) = IP(u(t)), u(t) = (f(t), g(t))$. Then $h'(a) = [DIP(f(a), g(a))][Du(a)] = (g(a), f(a)) * (f_1'(a), ...f_n'(a), g_1'(a)...g_n'(a))^t = \langle f'(a)^t, g(a)\rangle + \langle f(a), g'(a)^t\rangle$.

3. Put $h(t) = \langle f(t), f(t)\rangle = 1$ in the previous problem. Since $h(t)$ is a constant function, $\frac{dh}{dt} = 0 \implies \langle f'(t)^t, f(t)\rangle + \langle f(t), f'(t)^t\rangle = 2\langle f'(t)^t, f(t)\rangle = 0 \implies \langle f'(t)^t, f(t)\rangle = 0$ for all t.

4. Let $f(t) = t.|t|$ is not differentiable at 0.

2-14.

1. Let $g : E_i \times E_j \to \mathbb{R}^p, g(x,y) = f(a_1...a_{i-1}, x, a_{i+1}, ...a_{j-1}, y, a_{j+1}, ...a_k)$. This is clearly a bilinear map. Thus, $0 \leq lim_{(h_i,h_j)\to 0}\frac{|g(h_i,h_j)|}{|h|} \leq lim_{(h_i,h_j)\to 0}\frac{|g(h_i,h_j)|}{|(h_i,h_j)|} = 0$ (from 2-13, i) $\implies lim_{h\to 0}\frac{f(a_1,...h_i,...h_j,...a_k)}{|h|} = 0$.

2. $f(a_1+h_1, a_2+h_2, ...a_k+h_k) = f(a_1, ...a_k) + \sum_{i=1}^k f(a_1, ..h_i, ...a_k) + \sum_{1\leq j_1<j_2\leq k}^k f(a_1, ...h_{j1}, ...h_{j2}, ...a_k) +$ other terms. The last set of other terms will each have three or more entries of h_i in the function's argument. We denote them collectively by $R(h)$.
We claim that $lim_{h\to 0}\frac{|f(a_1+h_1,...a_k+h_k) - f(a_1,...a_k) - \sum_{i=1}^k f(a_1,..h_i,...a_k)|}{|h|} = 0$. This will show that $\lambda(h_1, ...h_k) = \sum_{i=1}^k f(a_1, ..h_i, ...a_k)$. From the above, we have $0 =$
$lim_{h\to 0}\frac{|f(a_1+h_1,...a_k+h_k) - f(a_1,...a_k) - \sum_{i=1}^k f(a_1,..h_i,...a_k) - \sum_{1\leq j_1<j_2\leq k}^k f(a_1,...h_{j1},...h_{j2},...a_k) - R(h)|}{|h|} \geq$
$lim_{h\to 0}\frac{|f(a_1+h_1,...a_k+h_k) - f(a_1,...a_k) - \sum_{i=1}^k f(a_1,..h_i,...a_k)|}{|h|} - lim_{h\to 0}\sum_{1\leq i<j\leq k}^k\frac{|f(a_1,...h_i,...h_j,...a_k)|}{|h|} - lim_{h\to 0}\frac{|R(h)|}{|h|}$.
The second term vanishes, by the previous part of this exercise. We can argue similarly for the third grouping of terms by holding one of the h_i components constant.
Thus, $Df(a_1, ...a_k)(x_1, ..x_k) = \sum_{i=1}^k f(a_1, ...x_i, ...a_k)$.

2-15.

1. We know that the determinant is a multilinear map. Thus, this result follows immediately from the above.

2. From the chain rule, we have $f'(t) = [Ddet(a_{ij}(t))]o[Da_{ij}(t)] = \sum_{i=1}^n \det\begin{pmatrix} a_1 \\ . \\ a_i' \\ . \\ a_n \end{pmatrix}$
where $a_i' = (a_{i1}', ...a_{in}')$.

3. Let $A = (a_{ji}), s = s_i = b = b_i$, where s, b are column vectors. Then, we have $As = b$. Cramer's rule yields $s_i = \frac{detA_i}{detA} = \frac{g_i(t)}{f(t)}$ (say). Then, $s_i'(t) = \frac{f(t)g_i'(t) - f'(t)g_i(t)}{(f(t))^2}$

2-16. Define $h : \mathbb{R}^n \to \mathbb{R}^n, h(x) = x$. Clearly, $fof^{-1} = h$. Thus, $D(fof^{-1})(a) = Dh(a) = f'(f^{-1}(a)) * (f^{-1})'(a)$. But $Dh(a) = I \implies (f^{-1})'(a) = [f'(f^{-1}(a)]^{-1}$.

2-17.

1. $\frac{\partial f}{\partial x} = yx^{y-1}, \frac{\partial f}{\partial y} = x^y ln(x)$

2. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 1$.

3. $\frac{\partial f}{\partial x} = sin(y)cos(xsin(y)), \frac{\partial f}{\partial y} = xcos(y)cos(xsin(y))$

4. $\frac{\partial f}{\partial x} = (sin(ysin(z))cos(x(sin(ysin(z))), \frac{\partial f}{\partial y} = cos(x(sin(ysin(z)))xcos(ysin(z))sin(z)$,
$\frac{\partial f}{\partial z} = cos(x(sin(ysin(z)))xycos(ysin(z))cos(z)$

5. $\frac{\partial f}{\partial x} = y^z x^{y^z-1}, \frac{\partial f}{\partial y} = zy^{z-1}x^y ln(x), \frac{\partial f}{\partial z} = x^{y+z}ln(x)$

6. $\frac{\partial f}{\partial x} = (y+z)x^{y+z-1}$, $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = x^{y+z}ln(x)$

7. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = z(x+y)^{z-1}$, $\frac{\partial f}{\partial z} = (x+y)^z ln(x+y)$

8. $\frac{\partial f}{\partial x} = ycos(xy)$, $\frac{\partial f}{\partial y} = xcos(xy)$

9. $\frac{\partial f}{\partial x} = cos(3)xcos(xy)sin^{cos(3)-1}(xy)$, $\frac{\partial f}{\partial y} = cos(3)ycos(xy)sin^{cos(3)-1}(xy)$

2-18.

1. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = g(x+y)$.

2. $\frac{\partial f}{\partial x} = g(x)$, $\frac{\partial f}{\partial y} = -g(y)$.

3. $\frac{\partial f}{\partial x} = yg(x,y)$. $\frac{\partial f}{\partial y} = xg(x,y)$.

4. $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = g(\int_a^y g)g(y)$.

2-19. Note that $f(1,y) = 1$ everywhere. Since it is constant, we have $D_2 f(1,y) = 0$.

2-20.

1. $f(x,y) = p(u,v) = uv, u = g(x), v = h(y)$.
   $\frac{\partial f}{\partial x} = \frac{\partial p}{\partial u}\frac{dg}{dx} + \frac{\partial p}{\partial v}\frac{dh}{dx} = vg'(x) + u * 0 = g'(x)h(y)$. By symmetry, we can also say $\frac{\partial f}{\partial y} = g(x)h'(y)$.

2. $f(x,y) = p(u,v) = u^v, u = g(x), v = h(y)$.
   $\frac{\partial f}{\partial x} = \frac{\partial p}{\partial u}\frac{dg}{dx} + \frac{\partial p}{\partial v}\frac{dh}{dx} = vu^{v-1}g'(x) + 0 = g(x)^{h(y)-1}g'(x)h(y)$.
   $\frac{\partial f}{\partial y} = \frac{\partial p}{\partial u}\frac{dg}{dy} + \frac{\partial p}{\partial v}\frac{dh}{dy} = 0 + u^v ln(u)h'(y) = g(x)^{h(y)}h(y)ln(g(x))$.

3. $\frac{\partial f}{\partial x} = g'(x)$, $\frac{\partial f}{\partial y} = 0$.

4. $\frac{\partial f}{\partial y} = g'(y)$, $\frac{\partial f}{\partial x} = 0$.

5. $f(x,y) = g(p), p = p(x,y) = x+y$.
   $\frac{\partial f}{\partial x} = \frac{dg}{dp}\frac{\partial p}{\partial x} = g'(x+y)$. Similarly, $\frac{\partial f}{\partial y} = g'(x+y)$.

2-21.

1. $\frac{\partial f(x,y)}{\partial y} = \frac{\partial \int_0^x g_1(t,0)dt}{\partial y} + \frac{\partial \int_0^y g_2(x,t)dt}{\partial y} = D_2 g(x,y)$ (fundamental theorem of calculus).

2. $f(x,y) = \int_0^x g_1(t,y)dt + \int_0^y g_2(0,t)dt$.

3. $f(x,y) = \frac{x^2+y^2}{2}$, $f(x,y) = xy$.

2-22. We apply the mean value theorem on one variable. So, $f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x,y_0)(y_1 - y_2)$ for some $y_0 \in [y_2, y_1]$. But we know that $\frac{\partial y}{\partial y} = 0 \forall (x,y) \implies f(x,y_1) = f(x,y_2) \forall x, y_1, y_2$. This shows that f is independent of the second variable. A similar argument can be used in case $D_1 f = 0$ to show that it is independent of the first variable; and if it is independent of both variables, it is just the constant function.

2-23.

1. Note that any two points in A can be connected by a sequence of lines parallel to one of the axes. Consider the endpoints of any one of the lines (parallel to the y-axis, without loss of generality) joining two arbitrary points in A. Applying the mean value theorem on it, we have $f(x_1,y) = f(x_2,y)$. Since this holds true for all the lines, the functional value will be equal at the two arbitrary points. Thus, f is constant.

2. Consider the function $f(x,y)=0$ on the second and third quadrants, x^2 on the first and $-x^2$ on the fourth. This is continuous on A (in fact, if defined on $\mathbb{R}^2$, it is discontinuous just on $\mathbb{R}^2\backslash A$) but clearly not independent of the second variable.

2-24.

1. $\frac{\partial f}{\partial y} = \frac{\partial(xy\frac{x^2-y^2}{x^2+y^2})}{\partial y} = -\frac{x(y^4+4x^2y^2-x^4)}{(x^2+y^2)^2}$. Evaluating at $(x,y)=(x,0)$, we have $\frac{\partial f}{\partial y} = x$. Also, $\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = 0$.
$\frac{\partial f}{\partial x} = \frac{\partial(xy\frac{x^2-y^2}{x^2+y^2})}{\partial x} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$. Evaluating at $(x,y)=(0,y)$, we have $\frac{\partial f}{\partial x} = -y$. Also, $\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = 0$.
2. $D_{1,2}f(0,0) = \frac{\partial D_2 f(x,0)(0)}{x} = 1 \neq -1 = \frac{\partial D1 f(0,y)(0)}{\partial y} = D_{2,1}f(0,0)$.

2-25. Clearly, for $x \neq 0, f'(x) = \frac{2e^{-x^{-2}}}{x^3}$. Its higher order derivatives will be of the form $\frac{p(x)e^{-x^{-2}}}{x^n}$. Now, $f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-2}}}{h} = \lim_{h\to 0}\frac{\frac{1}{h}}{e^{h^{-2}}} = \lim_{\to 0}\frac{h}{2e^{1\frac{1}{h^2}}} = 0$, using L'Hospital's rule. $f''(0) = \lim_{h\to 0}\frac{f'(h)}{h} = \lim_{h\to 0}\frac{1}{2e^{\frac{1}{h^2}}} = 0$. For higher order derivatives, we need only to apply L'Hospital's rule again.

2-27. Note that $A = \{x \in \mathbb{R}^2 : |x| \leq 1\}, B = \{x \in \mathbb{R}^3 : |x| = 1\} \implies B = g(A) \cup h(A)$. This completes the proof.

2-28.

1. $D_1F(x,y) = D_1f(u,v)k(y)g'(x) + D_2f(u,v)g'(x)$
$D_2F(x,y) = D_1f(u,v)k'(y)g(x) + D_2f(u,v)k'(y)$
2. $D_1F(x,y,z) = D_1f(u,v)k(y)g'(x+y)$
$D_2F(x,y,z) = D_1f(u,v)k(y)(x+y) + D_2f(u,v)h'(y+z)$
$D_3F(x,y,z) = D_2f(u,v)h'(y+z)$
3. $D_1F(x,y,z) = D_1f(u,v,w)yx^{-1} + D_3f(u,v,w)ln(z)z^x$
$D_2F(x,y,z) = D_1f(u,v,w)ln(y)x^y + D_2f(u,v,w)zy^{z-1}$
$D_3F(x,y,z) = D_2f(u,v,w)ln(y)y^z + D_3f(u,v,w)xz^{x-1}$
4. $D_1F(x,y) = D_1f(u,v,w) + D_2f(u,v,w)g'(x) + D_3f(u,v,w)D_1h(x,y)$
$D_2F(x,y) = D_3f(u,v,w)D_2h(x,y)$

2-29.

1. Follows from definition.
2. $D_{tx}f(a) = \lim_{h\to 0}\frac{f(a+htx)-f(a)}{h}$. Substituting $u = ht$, we have $D_{tx}(a) = \lim_{u\to 0}\frac{f(a+ux)-f(a)}{(\frac{u}{t})} = tD_xf(a)$.
3. $\lim_{h\to 0}|\frac{f(a+h)-f(a)-Df(a)h}{h}| = 0$. Replacing h with tx, we have $\lim_{t\to 0}|\frac{f(a+tx)-f(a)-tDf(a)x}{tx}| = 0 \implies D_xf(a) = Df(a)x$. Linearity follows from linearity of the derivative.

2-30. $D_xf(0,0) = \lim_{t\to 0}|\frac{f(tx)}{t}| = \frac{|tx|g(\frac{tx}{|tx|})}{|t|} = f(x)$. Thus, it exists.
If we let $x = (1,0), y = (0,1)$, then $D_xf(0,0) = f(1,0) = 0, D_yf(0,0) = f(0,1) = 0$, but $D_{x+y}f(0,0) = f(1,1) = \sqrt{2}g(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \neq 0$ necessarily.

2-32.

1. $\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2 sin(\frac{1}{h})}{h} = \lim_{h\to 0} h sin(\frac{1}{h}) = 0$. Thus, f is differentiable at 0 and $f'(0) = 0$.
   Elsewhere, $f'(x) = 2x sin(\frac{1}{x}) - cos(\frac{1}{x})$, which has no limit as $x \to 0$. Thus, f' is not continuous at 0.

2. Rewriting $\sqrt{x^2+y^2} = |z|$, the argument runs unchanged.

2-33. Continuity of $D_j f$ at a is used to infer that, as $h \to 0 \equiv c_j \to a, D_j f(c_j) \to D_j f(a)$.
Supposing $D_1 f$ is not continuous at a, we may still apply the same argument for the remaining components. Then we have $\lim_{h\to 0}\frac{|f(a+h)-f(a)-\sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(a)h_j|}{||h||} \leq \lim_{h\to 0}\frac{|f(a+h_1)-f(a)-D_1 f(a)h_1|}{||h||} + \sum_{j=2}^{n}\frac{|D_j f(c^j)-D_j f(a)||h_j|}{||h||} = \lim_{h\to 0}|Df_1(a) - Df_1(a)|\frac{|h_1|}{||h||} = 0$, and we are done.

2-36. The inverse function theorem tells us that for any a, there is an open set $V \subset f(A)$ containing $f(a)$. Thus, we have found an open ball around every point $x \in f(A)$ such that $B \subset f(A)$. It follows that $f(A)$ is open. The same argument on the restriction of f to B works to show that $f(B)$ is open. It follows from the inverse function theorem that an inverse exists and is differentiable at each point $y \in f(A)$. Since we are given the existence of a global inverse, the result follows.

2-38.

1. Suppose $f(x_1) = f(x_2), x_1 \neq x_2$. We know that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). Thus, from the mean value theorem, $f(x_2) - f(x_1) = f'(a)(x_2 - x_1) = 0 \implies f'(a) = 0$ for some $a \in (x_1, x_2)$, contradicting the fact that $f'(x) \neq 0 \forall x \in \mathbb{R}$. Hence, proved.

2. $Df(a,b) = \begin{pmatrix} e^x cos(y) & -e^x sin(y) \\ e^x sin(y) & e^x cos(y) \end{pmatrix}$

   det $f'(x,y) = e^{2x}(cos^2 y + sin^2 y) = e^{2x} \neq 0 \forall (x,y) \in \mathbb{R}^2$.
   That f is not injective is clear from the fact that $f(x,y) = f(x, y + 2n\pi)$.

2-39. For $x \neq 0, f'(x) = \frac{1}{2} + 2x sin(\frac{1}{x}) - cos(\frac{1}{x})$. Note that its limit as x tends to 0 does not exist due to the fluctuating cosine. However, $f'(0) = lim_{h\to 0}\frac{\frac{h}{2}+h^2 sin(\frac{1}{h})}{h} = \frac{1}{2}$. The function is differentiable everywhere, but its derivative is not continuous at $x = 0$.
This is not invertible around $x = 0$ because in any neighbourhood around the origin, we can find a point at which the derivative vanishes.

2-40. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n : f^i(t, \mathbf{s}) = \sum_{j=1}^{n} a_{ji}(t) s_j - b_i(t) = 0$. Note that $\frac{\partial f^i}{\partial x_j} = a_{ij}(t)$
Thus, by the implicit function theorem, there is a function $s(t)$ such that $f(t, s(t)) = 0$, and $s(t)$ is differentiable.

2-41.

1. Note that $g_x'(y) = \frac{\partial f(x,y)}{\partial y}$. Thus, $\frac{\partial f(x,y)}{\partial y} = 0$ at $(x, c(x))$. Also, $\frac{\partial^2 f(x,y)}{\partial y^2} \neq 0$. Thus, we can apply the implicit function theorem on $D_2 f(x,y)$ to get differentiability of c. By the chain rule, we have $D_1(D_2 f(x, c(x))) = D_1(D_2 f(x, c(x))) + D_2(D_2 f(x, c(x)) c'(x) = 0 \implies c'(x) = \frac{-D_{2,1} f(x,c(x))}{D_{2,2} f(x,c(x))}$.

2. The first follows from the above result, with $y = c(x)$. Also, by definition, $y = c(x) \implies D_2 f(x,y) = 0$.

3. $\frac{\partial f(x,y)}{\partial y} = xlogy - logx = 0 \implies y = x^{\frac{1}{x}}$. $D_{2,2} = \frac{x}{y} > 0 \implies$ this is the minima we want.
   However, $\frac{1}{3} \leq x^{\frac{1}{x}} \leq 1$ only in the range $[a, 1]$ for some $a > \frac{1}{2}$.

- In the range $[\frac{1}{2}, a)$, the quantity will be minimized by $y = \frac{1}{3}$. $f(x, \frac{1}{3}) = -x(\frac{1+ln3}{3}) - \frac{1}{3}lnx$ is maximized at $x = \frac{1}{2}$. $f(\frac{1}{2}, \frac{1}{3}) = \frac{ln(\frac{4}{3e})}{6}$.
- In the range $[a, 1)$, we wish to maximize $f(x, x^{\frac{1}{x}}) = -x^{1+\frac{1}{x}}$, clearly maximized at $x = a$. It equals $-\frac{a}{3}$.
- In $[1, 2]$, it will be minimized by $y = 1$. $f(x, 1) = -x - lnx$. This is a decreasing function, maximum at $x = 1$. $f(1, 1) = -1$.

Comparing the three, we see that the required point is $(\frac{1}{2}, \frac{1}{3})$.