Analysis-III

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Spivak's Calculus on Manifolds, Chapter I

1-1. $|x|^2 = \sum_{i=1}^{n}(x_i)^2 = \sum_{i=1}^{n}(|x_i|)^2 \leq (\sum_{i=1}^{n}|x_i|)^2 \implies |x| \leq \sum_{i=1}^{n}|x_i|$.

1-2. It holds when x, y are linearly dependent and $\langle x, y\rangle \geq 0$.

1-3. $|x-y| = |x+(-y)| \leq |x| + |-y| = |x| + |y|$. Equality holds when x, y are linearly dependent and $\langle x, y\rangle \leq 0$.

1-4. $|x-y|^2 = \sum_{i=1}^{n}(x_i - y_i)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - 2\sum_{i=1}^{n} x_i y_i \geq |x|^2 + |y|^2 - 2|x||y| = (|x|-|y|)^2 = ||x|-|y||^2 \implies ||x|-|y|| \leq |x-y|$.

1-5. $|z-x| = |(z-y)+(y-x)| \leq |z-y| + |z-x|$. This says that the sum of two side-lengths of a triangle cannot be less than the length of the third.

1-7.

1. Suppose T is inner product preserving, that is, $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^n$. Setting $x = y$, we have $\langle T(x), T(x)\rangle = |T(x)|^2 = \langle x, x\rangle = |x|^2 \implies |T(x)| = |x|$, that is, T is norm preserving.
   Next, suppose T is norm preserving. $\langle Tx, Ty\rangle = \frac{|Tx+Ty|^2 - |Tx-Ty|^2}{4} = \frac{|T(x+y)|^2 - |T(x-y)|^2}{4} = \frac{|x+y|^2 - |x-y|^2}{4} = \langle x, y\rangle$, and we are done.
2. Since T is norm preserving, $|T(x)| = 0 \implies |x| = 0 \implies x = 0$. Since the kernel of T consists of the null vector alone, it is injective.
   Now, T^{-1} is defined only on the image of T. Thus, we can write $|T^{-1}(y)| = |T^{-1}(T(x))| = |x| = |T(x)| = |y|$ for some x all y, and we are done.

1-8.

1. We know that, if T is norm preserving, T is inner product preserving. Thus, $\angle(Tx, Ty) = arccos(\frac{\langle Tx, Ty\rangle}{|Tx||Ty|}) = arccos(\frac{\langle x, y\rangle}{|x||y|}) = \angle(x, y)$.
2. Suppose $|\lambda_i| = k$ for all i. Then, $\angle(Tx_i, Tx_j) = arccos(\frac{\langle Tx_i, Tx_j\rangle}{|Tx_i||Tx_j|}) = arccos(\frac{k^2\langle x_i, x_j\rangle}{k^2|x_i||x_j|}) = arccos(\frac{\langle x, y\rangle}{|x||y|}) = \angle(x, y)$. Thus, T is angle-preserving.

1-9. $T(x, y) = (xcos\theta + ysin\theta, -xsin\theta + ycos\theta)$. It suffices to show (by an above exercise) that T is norm preserving. $|T(x, y)|^2 = (xcos\theta + ysin\theta, -xsin\theta + ycos\theta)(xcos\theta + ysin\theta, -xsin\theta + ycos\theta) = x^2cos^2\theta + y^2sin^2\theta + 2xycos\theta sin\theta + x^2sin^2\theta + y^2cos^2\theta - 2xycos\theta sin\theta = x^2 + y^2 = |(x, y)|^2$.
Let $x = (a, b)$. $\langle x, Tx\rangle = (a, b)(acos\theta + bsin\theta, -asin\theta + bcos\theta) = cos\theta|(a, b)|^2$. Thus, $arccos(\frac{\langle x, Tx\rangle}{|x||Tx|}) = arccos(\frac{cos\theta|(a,b)|^2)}{|(a,b)|^2} = \theta = \angle(x, Tx)$.

1-10. Let $T = (a_1, a_2, ...a_n)^T$, where each a_i is a 1 x m row vector. Then, $Th = (\langle a_1, h\rangle, ...\langle a_n, h\rangle)^T.|Th| = \sqrt{\sum_{i=1}^{n}\langle a_i, h\rangle^2} \leq \sqrt{\sum_{i=1}^{n}|a_i|^2|h|^2} = |h|\sqrt{\sum_{i=1}^{n}|a_i|^2}$. Thus, the inequality is satisfied for $M =$

$\sqrt{\sum_{i=1}^{n}|a_i|^2}$.

1-11. $\langle (x,z),(y,w)\rangle = \langle (x_1,...x_n,z_1,...z_m),(y_1,...y_n,w_1,...w_m)\rangle = \sum_{i=1}^{n} x_iy_i + \sum_{i=1}^{m} z_iw_i = \langle x,y\rangle + \langle z,w\rangle$ (where $x_i,y_i,z_i,w_i \in \mathbb{R}$).
$|(x,z)| = |(x_1,...x_n,z_1,...z_m)| = |\sqrt{x_1^2+...x_n^2+z_1^2+...z_m^2} = \sqrt{|x|^2|+|z|^2}$.

1-12. $T(x) = T(y) \implies \phi_x = \phi_y \implies \langle x,z\rangle = \langle y,z\rangle$ for all $z \in \mathbb{R}^n$. Setting $z = (1,0,...0)$, we have $x_1 = y_1$. We can match x, y componentwise in this manner to get $x = y$. Thus, T is an injective map.

1-13. $|x+y|^2 = \langle x+y,x+y\rangle = \langle x,x\rangle + \langle y,y\rangle + 2\langle x,y\rangle = |x|^2+|y|^2$, since the last term vanishes due to orthogonality.

1-14. Let $U = \bigcup_{\alpha\in\Delta} U_\alpha$, where each U_α is open. Suppose $x \in U$. Then, $x \in U_{\alpha 0}$ for some α_0. By assumption, since $U_{\alpha 0}$ is open, there exists an open ball around x such that $B(x) \subset U_{\alpha 0} \subset U$. Thus, U is open.
Next, let $U = \bigcap_{i\in S} U_i$, where each U_i is open and S is some finite subset of $\mathbb{N}$. Suppose $x \in U$. Then, $x \in U_i$ for all $i \in S$. In every set U_i, there is an open ball around the point, $B_{ri}(x) \subset U_i$. Set $R = min\{r_i, i \in S\}$. $B_R(x)$ will then be a subset of all the sets U_i. Thus, $B_R(x) \subset U$, and we are done.
Each of the sets $(-\frac{1}{n}, 1)$ is open, but their intersection (as n varies over all the naturals) is $[0,1)$, which is not.

1-15. Define $S = \{x \in \mathbb{R}^n : |x-a| < r\}$, and pick an arbitrary point $p \in S$. Thus, $|p-a| = r-\epsilon < r, \epsilon > 0$. Consider the open ball $B(p, \frac{\epsilon}{2})$. It is clear that $B \subset S$: For any $q \in B, |q-a| < |q-p|+|p-a| \leq \frac{\epsilon}{2}+r-\epsilon = r-\frac{\epsilon}{2} < r$. Thus, we have found an open ball around p which is a subset of S. Since p was taken arbitrarily, this shows that S is open.

1-16.

1. Let $S = \{x \in \mathbb{R}^n : |x-0| \leq 1\}$ The interior is the set $\{x \in \mathbb{R}^n : |x| < 1\}$. The exterior is the set $\{x \in \mathbb{R}^n : |x| > 1\}$. The boundary is the set $\{x \in \mathbb{R}^n : |x| = 1\}$.
   We only need to prove the third claim. The first two then follow from arguments analogous to the one in the previous problem.
   Pick any arbitrary point p in the third set, and take an open ball around it, $B(p,\delta), delta > 0$. Consider the line joining the origin and p, $\{pt : 0 \leq t \leq 1\}$. The distance between the origin and some point on this line is given by $|pt-0| = |p||t| = |t|$; and the distance between p and some point on this line is given by $|pt-p| = |p||1-t| = |1-t|$. Now, if $|t| < 1, |1-t| < \delta$, this point lies in S. Alternatively, if $|t| > 1, |1-t| < \delta$, this point lies in the complement of S. Thus, B is not a subset of S or its complement, and p is a boundary point.
2. The interior is $\emptyset$. The exterior is the set $\mathbb{R}^n \backslash \{x \in \mathbb{R}^n : |x| = 1\}$. The boundary is the set $\{x \in \mathbb{R}^n : |x| = 1\}$. The argument is analogous to the above.
3. Let S denote the given set. The interior is $\emptyset$ (from the Archimedean property of the reals). The set of boundary points is S, since they are not in the interior, and trivially not in the exterior. Thus, the exterior is $\mathbb{R}^n \backslash S$.

1-19. Let $x \in [0,1]$ be irrational. By the Archimedean property, any open ball around it will have at least one rational number. Thus, it is a limit point; since A is closed, we conclude that $x \in A$. This shows that $A \subseteq [0,1]$.

1-20. If a compact subset is unbounded, we could let $B(0,n), n \in \mathbb{N}$ be an open cover (since the union, in fact, equals the space), and it would have no finite subcover (since the set is unbounded), contradicting compactness. Thus, every compact set is bounded.
Next, we wish to show that if a set K is compact, it is closed. Consider a point $y \in \mathbb{R}^n \backslash K, x \in K$,

and take open neighbourhoods V_y, U_x for both such that $V_y \cap U_x = \emptyset$. Now, $\bigcup_{x \in K} U_x$ is an open cover of $K \implies$ it has a finite subcover, $U_{x1}, U_{x2}, ...U_{xm}$. Consider the corresponding intersection of $V = \bigcap_{i=1}^{m} V_{yi}$. Clearly, $V \cap K = 0$, and so $V \subset \mathbb{R}^n \backslash K$ and is an open neighbourhood of y. Thus, $\mathbb{R}^n \backslash K$ is open, and it follows that K is closed.

1-21.

1. By assumption, x is not a limit point of A. This means that some open ball around $x, B(x,d)$ is disjoint from A. By definition, this means $|y - x| \geq d$ for all $y \in A$.

2. Pick a $b \in B \implies b \notin A$. By the above, this means that some open ball around $x, B(x,d)$ is disjoint from A; and so, so is $C_b = B(b,d) \cap B$. Thus, $\exists d > 0 : |y - x| \geq d$ for all $x \in C$. We can find such a ball for each $b \in B$. Moreover, since these balls form an open cover and B is compact, we can find a finite subcover of the same. The minimum of the radii of these finitely many open balls yields the required d. ($x \in B \implies x \in C_p = B(p,r) \cap U$ for some $p \in B \implies |y - x| \geq r \geq d = r_{min}$.)

3. The sets in $\mathbb{R}^2$ defined by $A = \{(x, \frac{1}{x}), x \in \mathbb{R}^+\}$, $B = \{(0,x), x \in \mathbb{R}^+\}$ provide the required counterexample.

1-22. Consider an open cover A_α of C such that $\overline{A_\alpha} \subset U$ for each one. Since C is compact, this has a finite subcover, $\{A_1, ...A_n\}$. Consider the compact set $D = (\overline{A_1} \cup ... \cup \overline{A_n}) \subset U$. By construction, the interior of this is a superset of C. Thus, this is the required set.

1-23. First, suppose $\lim_{x \to a} f^i(x) = b^i, i = 1, ...m$. So, $\forall \epsilon > 0 \exists \delta > 0$ such that $|x - a| < \delta \implies |f^i(x) - b^i| < \epsilon$. Now, $|f(x) - b| = \sqrt{(f^1(x) - b^1)^2 + (f^2(x) - b^2)^2 + ...(f^m(x) - b^m)^2} < \epsilon$ if $|x - a| < \frac{\delta}{\sqrt{m}}$.
For the converse, suppose $\forall \epsilon > 0 \exists \delta > 0$ such that $|x - a| < \delta \implies \sqrt{(f^1(x) - b^1)^2 + (f^2(x) - b^2)^2 + ...(f^m(x) - b^m)^2} = |f(x) - b| < \epsilon$. Since $|f^i(x) - b^i| \leq |f(x) - b|$, we have the desired implication.

1-24. The proof is the same as above, except we replace b with $f(a)$.

1-25. $|x - a| < \delta = \frac{\epsilon}{M} \implies |T(x) - T(a)| = |T(x - a)| \leq M|(x - a)| < M\frac{\epsilon}{M} = \epsilon$. Thus, we have found a δ, given an ϵ.

1-26.

1. A straight line through the origin is given by the equation $y = mx$. If $m \leq 0$, the whole line lies in $\mathbb{R}^2 - A$. Consider $m > 0$. In this case, $\{x \in (-m, m), y = mx\}$ gives the required interval.

2. Clearly, $f(0,0) = 0$ but for the sequence $x_i = (\frac{1}{i}, \frac{(\frac{1}{i})^2}{2})$ we have $lim_{i \to \infty} f(x_i) = 1, x_i \to 0$. Thus, $f(x)$ is not continuous at $(0,0)$.
On the other hand, $g_h(t) = f(t\mathbf{h})$ is the restriction of f to some particular straight line through the origin. From the above, it follows that g is continuous.

1-27. This is clearly a continuous function (it can be seen as the composition of $g(x) = x - a, h(x) = |x|$, both continuous functions).
Consider the open set $U = (-\infty, r) \in \mathbb{R}$. Then, there must be an open set $V \in \mathbb{R}^n$ such that $f^{-1}(U) = V$. But by definition, $V = \{x \in \mathbb{R}^n : |x - a| < r\}$. Hence, proved.

1-28. Since $A \subset \mathbb{R}^n$ is not closed, we can pick a limit point $l \notin A$. Now, the function $f : A \to \mathbb{R} : f(y) = \frac{1}{|y - l|}$ is clearly continuous on A, since it is the composition of continuous functions ($k(x) = x - l, g(x) = \frac{1}{x}, h(x) = |x|$, where the former is continuous because 0 is not in its domain, since by assumption $l \notin A$).

It remains to be shown that f is unbounded. Since l is a limit point of A, for any $M \in \mathbb{R}$, we can find $y \in$ such that $|y - l| < M \implies \frac{1}{|y-l|} > M$. This completes the proof.

1-29. We know that the image of a continuous function is compact if its pre-image is compact. Furthermore, we know that a compact set contains its maximum and minimum. (Suppose not, i.e. suppose $\sup f(A) \notin f(A)$. We know $\sup A$ exists since $\mathbb{R}$ is complete and the set is bounded. But this would contradict closedness of $f(A)$, since the supremum is a limit point of the set.) This proves the claim.