Sequences and series of functions

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Answer 1. Given: $\forall\epsilon > 0\ \exists N_0 \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon\ \forall n, m > N_0 \forall x$, and for each n, there exists an M'_n such that $|f_n(x)| \leq M'_n\ \forall x$.
To prove: $|f_n(x)| \leq M\ \forall n \forall x$ for some M.
We have for all $n, m > N_0$, for all x, $|f_n(x)| < \epsilon + |f_m(x)|$. Set $M = \max\{M_1, M_2, ...M_{n-1}, \epsilon + M_m\}$, where M_k is the bound for $|f_k(x)|$. This gives us the required uniform bound.

Answer 2. By definition, we have $\forall\epsilon > 0\ \exists N_\epsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon\ \forall n > N_\epsilon\ \forall x$, and similarly for g_n and g.
Suppose $|f_n(x) - f(x)| < \frac{\epsilon}{2}\ \forall n > N_f$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2}\ \forall n > N_g$ for all x.
Then, $|f_n(x)-f(x)|+|g_n(x)-g(x)| \geq |f_n(x)-f(x)+g_n(x)-g(x)| = |(f_n(x)+g_n(x))-(f(x)+g(x))| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\ \forall n > \max\{N_f, N_g\}$. (Note that we have used the triangle inequality here.) Since N_f, N_g depended only on ϵ, so does their maximum. Thus, we conclude that $\{f_n + g_n\}$ converges uniformly to $f + g$.
Next, suppose that the functions are bounded. This means that $f_n(x) \leq M$ (and so also, $f(x) \leq M$) for some $M \in \mathbb{R}$ for all n and x, and similarly for $g_n(x)$.
Suppose $|f_n(x) - f(x)| < \frac{\epsilon}{2M}\ \forall n > N_f$ and $|g_n(x) - g(x)| < \frac{\epsilon}{2M}\ \forall n > N_g$ for all x.
Then, $|f_n(x)g_n(x)-f(x)g(x)| \leq |f_n(x)g_n(x)-f_n(x)g(x)|+|f_n(x)g(x)-f(x)g(x)| \leq M|f_n(x)-f(x)|+M|g_n(x)-g(x)| < M\frac{\epsilon}{2M}+M\frac{\epsilon}{2M} = \epsilon$ for all $n > \max\{N_f, N_g\}$. This shows that the uniform converges uniformly.

Answer 3. We define $\{f_n(x)\} = \{g_n(x)\} = x + \frac{1}{n}$.
Define $F(x) = x, x \in \mathbb{R}$. Then, $|f_n(x) - F(x)| = |\frac{1}{n}| < \epsilon \implies n > \frac{1}{\epsilon}$. Thus, $|f_n(x) - F(x)| < \epsilon$ $\forall n > N_0$, where $N_0 = 1 + [\frac{1}{\epsilon}]$. Since N_0 is clearly independent of x, we conclude $\{f_n\}$ converges uniformly to F.
$\{f_n^2\} = x^2 + 2\frac{x}{n} + \frac{1}{n^2}$. Define $F(x) = x^2, x \in \mathbb{R}$. Then, $|f_n^2(x) - F(x)| = |2\frac{x}{n} + \frac{1}{n^2}| < \frac{2x+1}{n} < \epsilon$ for all $n > N_0$, where $N_0 = 1 + [\frac{2x+1}{\epsilon}]$. Thus, it converges pointwise.
Suppose now there existed an $N_\epsilon \in \mathbb{N}$ such that $|2\frac{x}{n} + \frac{1}{n^2}| < \epsilon\ \forall n > N_\epsilon\ \forall x$. But we see that upon putting $n = N_\epsilon + 1, x = \epsilon\frac{N_\epsilon+1}{2}$, we get $\epsilon + \frac{1}{N_\epsilon+1} < \epsilon$, a contradiction. We conclude that $\{f_n^2\}$ does not converge uniformly to F on $\mathbb{R}$.

Answer 4. The series of functions $f_n(x) = \frac{1}{1+n^2x}$ converges pointwise and absolutely for all $x \in \mathbb{R}\backslash\{0, -\frac{1}{n^2}|n \in \mathbb{N}\}$.
That it does for $x \in \mathbb{R}^+$ follows from the convergence of the series $\frac{1}{n^2}$. That it does for $x \in \mathbb{R}^-\backslash\{-\frac{1}{n^2}|n \in \mathbb{N}\}$ follows from the fact that it converges absolutely (since $|\frac{1}{1+n^2x}| \leq \frac{4}{3n^2|x|}$ for $n^2 \geq \frac{4}{|x|}$, and the RHS converges).
It does not converge for $x = 0, x = \{-\frac{1}{n^2}|n \in \mathbb{N}\}$ because we have either $f_n(x) = 1$, which diverges, or $f_n(x) = 1\frac{1}{1-(n^2\frac{1}{n^2})}$ for one term in the series, which is undefined.

We claim that the series converges uniformly on the interval $E = (c, \infty)$ with $c > 0$.
It is clear that $|f_n(x)| \leq \frac{1}{1+n^2c}$. Also, the series $\frac{1}{1+n^2c}$ converges (this follows from the convergence of the series $\frac{1}{n^2}$. Thus, it follows from Weierstrass' M test that the given series converges in the interval (c, ∞).
It does not, however, converge uniformly on $(0, \infty)$. To see this, for an $N_0\mathbb{N}$, we can set $n = N_0+1, m = N_0, x = \frac{1}{(N_0+1)^2}$ and see that it fails to be uniform Cauchy.
We claim that the series converges uniformly also on the interval $E = (-\infty, -1)$. Notice that $sup_{x\in E}|f_n(x)| \leq |\frac{1}{n^2-1}|$. Thus, it follows from Weierstrass' M test that the given series converges uniformly in the interval $(-\infty, -1)$.
It also converges uniformly on any interval of the form $(-\frac{1}{k^2}, -\frac{1}{(k+1)^2}), k \in \mathbb{N}$. One may apply the M-test here as well with $M_n = \frac{(k+1)^2}{n^2-(k+1)^2}$.
It is now evident that f is continuous but not bounded where the series converges.

Answer 5. $\{f_n(x)\}$ converges to 0 for all x. This is because, for a given $x > 0$, we can always find N_0 such that $\frac{1}{n} < x$ for all $n > N_0$, and for $x \leq 0$, we always have $x < \frac{1}{n+1}$. Since $F(x) = 0$ is a continuous function, we have shown that $\{f_n(x)\}$ converges to a continuous function.
However, the sequence does not converge uniformly. For any given positive integer N_0, we can always find $x = \frac{1}{(N_0+1)+\frac{1}{2}}$ such that $|f_{N_0+1}(x)| = sin^2\frac{\pi}{x} = 1 > \epsilon = \frac{1}{2}$.
From the above considerations, it is clear that $\sum f_n(x) = \sum|f_n(x)| = sin^2\frac{\pi}{x}$ for $0 < x < 1$, and 0 elsewhere, allowing us to conclude that it converges absolutely. However, since the function the series converges to is discontinuous at 0, we conclude that the function does not converge uniformly on any interval which has 0 as a limit point.

Answer 6. $\sum_{n=1}^{\infty}|f_n(x)| = \sum_{n=1}^{\infty}(\frac{x^2}{n^2} + \frac{1}{n})$. We know that $\frac{1}{n}$ does not converge. Thus, the series does not converge absolutely for any x (by comparison test).
Consider the interval $E = [a, b]$. Now, $\sum_{n=1}^{\infty}|f_n(x)| = \sum_{n=1}^{\infty}(-1)^n(\frac{x^2}{n^2} + \frac{1}{n}) = \sum_{n=1}^{\infty}(x^2\frac{(-1)^n}{n^2}) + (\frac{(-1)^n}{n}) = x^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ (since both the series converge).
Since the second series is also independent of x, we can say immediately that it converges uniformly.
Let us call the first series $h_n(x)$. In this case, we have $|h_n(x)| \leq \frac{k^2}{n^2}$, where $k = max\{|a|, |b|\}$. But also, we know that the series $\frac{k^2}{n^2}$ converges. Thus, by the M-test, $h_n(x)$ converges uniformly.
The sum of two uniformly convergent series converges uniformly. Hence, proved.

Answer 7. We claim that $\{f_n(x)\}$ converges uniformly to $f(x) = 0$.
It is clear that for all $\epsilon > 0$, we have $|\frac{x}{1+nx^2}| < \epsilon$ for all $n > N_0$, where $N_0 = \max\{1, 1 + [\frac{x}{\epsilon} - 1]\}$. Thus, we have pointwise convergence.
Note that $sup_{x\in\mathbb{R}}|f_n(x) - f(x)| = \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$ ($f_n'(x) = 0$ at $x = \frac{1}{\sqrt{n}}$). We conclude that the function converges uniformly.
$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \to 0$ as $n \to \infty$ for $x \neq 0$, and $= 1$ if $x = 0$. Furthermore, $f'(x) = 0$. Thus, we have the required equation.

Answer 8. It is clear that $\sup_{x\in(a,b)}|f_n(x)| \leq |c_n|$. Furthermore, we know that $\sum|c_n|$ converges. From the M-test, it follows that the series $f_n(x)$ converges uniformly in the given interval.
We now prove continuity at $x \in (a, b) \neq x_n \forall n \in \mathbb{N}$:
$s_n(x) = k_n \leq \sum_{k=1}^{n} c_k$. (Some of the points in the sequence before x_n may have been $> x$.) Let $\min\{|x - x_k|\}_{k=1}^{n} = k > 0$ (by assumption). For every $\epsilon > 0$, there exists a $\delta > 0$ such that $|y - x| < \delta \implies |s_n(y) - k_n| < \epsilon$. This is true, because one can pick any $\delta < k$, making the

RHS $0 < \epsilon$. Thus, $s_n(x)$ is continuous at the assumed x.
We know that a series of continuous functions, if it converges uniformly, converges to a continuous function. Thus, from the continuity of each $\{s_n(x)\}$, the continuity of $f(x)$ for every $x \neq x_n$ has been proven.

Answer 9. For every $\delta > 0$, there exists an $N_\delta \in \mathbb{N}$ such that $|x_n - x| < \delta$ for all $n > N_\delta$. For every $\frac{\epsilon}{2} > 0$, there exists an $N_\epsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $n > N_\epsilon$, for all x.
Furthermore, since the function is continuous, for every $\frac{\epsilon}{2} > 0$, there exists a $\delta' > 0$ such that $|x_n - x| < \delta' \implies |f_n(x_n) - f_n(x)| < \frac{\epsilon}{2}$.
$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n >$max$\{N_{\delta'}, N_\epsilon\}$.
The converse is not true in general. Let $f_n(x) = \frac{x}{n}, x_n = x$ on $\mathbb{R}$. Then $lim_{n\to\infty} f_n(x_n) = 0$, but it does not converge uniformly.

Answer 20. It is clear that this condition means $\int_0^1 f(x)p(x)dx = 0$, where $p(x)$ is any polynomial.
By the Weierstrass approximation theorem, there exists a sequence of polynomials $\{p_n(x)\}$ such that the uniform limit of it on $[0,1]$ is $f(x)$. Therefore, $lim_{n\to\infty} p_n(x)f(x) = f(x) lim_{n\to\infty} p_n(x) = f^2(x)$ uniformly on $[0,1]$.
By the uniform convergence theorem $\int_0^1 f^2(x)dx = lim_{n\to\infty} \int_0^1 p_n(x)f(x) = 0 \implies f^2(x) = 0 \implies f(x) = 0$. Hence, proved.