Differentiation

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February 2021

Answer 1. $|f(x)-f(y)| \leq (x-y)^2 = |x-y|^2 \implies |\frac{f(y)-f(x)}{y-x}| = \phi(y) \leq |(y-x)| \implies lim_{y\to x}\phi(y) \leq lim_{y\to x}|(y-x)| = 0 \implies f'(x) = 0$ everywhere. It is easy to see that the mean value theorem implies that it is a constant function.

Answer 2. Given: $f'(x) > 0$ on (a,b) and g is the inverse function of f.
By the mean value theorem, $\exists c \in (a,x), x \geq b$ such that $f(x)-f(a) = (x-a)f'(c)$. But since $f'(c) > 0$ and $x > a$, we have $f(x) > f(a) \implies f$ is strictly increasing on (a,b).
We know that, if f is a continuous bijection, so is its inverse. $f(x) \in (\alpha,\beta) \implies x \in (a,b)$ such that $x = g(f(x))$. Consider $g'(f(x)) \lim_{h\to 0} \frac{g(f(x)+h)-g(f(x))}{h}$. Suppose now that $g(f(x)+h) = x+k$. Then, we have $lim_{h\to 0}\frac{x+k-x}{h} = lim_{h\to 0}\frac{k}{h} = lim_{h\to 0}\frac{k}{f(x+k)-f(x)}$.
But now note that, by continuity of g, $h \to 0 \implies k \to 0$. Thus, our limit becomes $lim_{k\to 0}\frac{k}{f(x+k)-f(x)} = \frac{1}{f'(x)}$.
Hence, proved.

Answer 3. We know that $f'(x) \neq 0$ on $\mathbb{R} \implies f$ is one-one. Therefore, we set $f'(x) = 1+\epsilon g'(x) \neq 0$. If $g'(x) > 0$, this is ensured, since $\epsilon > 0$. $-M < -z = g'(x) < 0 \implies 1-\epsilon z \neq 0 \implies \epsilon \neq \frac{1}{z}$. If $f'(x)$ is ever > 0, we want to ensure it always remains $> 0 \implies \epsilon < \frac{1}{M}$. Otherwise, ϵ can be any real number.

Answer 4. Consider the polynomial $p(x) = C_0x + \frac{C_1x^2}{2} + ...$ Clearly, $p(0) = p(1) = 0 \implies p'(x) = 0$ for some $x \in (0,1)$ (by Rolle's theorem). Since $p'(x)$ is the given polynomial, this completes the proof.

Answer 5. By the mean value theorem, there exists an $x_0 \in (x, x+1)$ such that $f(x+1)-f(x) = f'(x_0)$. But also, for all $\epsilon > 0$ there exists an r such that $|f'(x_0)| < \epsilon$ for all $x_0 > r \implies$ for all $\epsilon > 0$ there exists an r such that $|f(x+1)-f(x)| < \epsilon$ for all $x > r \implies$ for all $\epsilon > 0$ there exists an r such that $|g(x)-0| < \epsilon$ for all $x > r$. Hence, proved.

Answer 6. $g(x) = \frac{f(x)}{x} \implies g'(x) = \frac{xf'(x)-f(x)}{x^2}$. We need to show that $xf'(x)-f(x) > 0$ for $x > 0$. Consider any interval $(0,a), a > 0$. By the mean value theorem, $\frac{f(a)}{a} = f'(c)$ for some $c \in (0,a)$. But since $f'(x)$ is increasing, $f'(a) > f'(c) \implies f'(a) > \frac{f(a)}{a} \implies xf'(x) > f(x)$ for all $x > 0$. Hence, proved.

Answer 7. $lim_{t\to x}\frac{f(t)}{g(t)} = lim_{t\to x}\frac{f(t)-f(x)}{g(t)-g(x)}$ (since $f(x) = g(x) = 0$) $= lim_{t\to x}\frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{lim_{t\to x}\frac{f(t)-f(x)}{t-x}}{lim_{t\to x}\frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}$.

Answer 9. Define $\varphi(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} & x \neq 0 \\ 3 & x = 0 \end{cases}$
It is clear from what is given that φ is a continuous function. Thus, by the definition of the derivative, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 3$.

Answer 12. For $x > 0$, $f'(x) = 3x^2, f''(x) = 6x$. For $x < 0, f'(x) = -3x^2, f''(x) = -6x$. Since left-hand and right-hand derivative are equal at 0, the limit and the derivative exists.
However, $f'''(x) = 6, x > 0$ and $-6, x < 0$. Since this has no limit at 0, it does not exist.

Answer 13.

1. To show that f is continuous at 0, we want for all $\epsilon > 0$ a $\delta > 0$ such that $|x| \leq \delta \implies |f(x)| = |x^a sin(\frac{1}{|x|^c})| \leq |x^a| \leq \epsilon$. For this, we observe that for $a > 0$, $|x^a| = |x|^a$ and thus set $\delta \leq \epsilon^{\frac{1}{a}}$.
   To prove that the function is discontinuous for $a \leq 0$, we set $\frac{1}{x} = ((2n+1)\frac{\pi}{2})^{\frac{1}{c}}$. Then $|f(x)| = |\frac{2}{(2n+1)\pi}|^{\frac{-a}{c}} sin(|(2n+1)\frac{\pi}{2}|) > \frac{1}{2}$ for $n > [\frac{1}{\pi}(2)^{\frac{-c}{a}}]$. Let x_1 be the corresponding number. Further, for any $\delta > 0$, let us consider an $x_2 < \delta$. Choosing the minimum of x_1, x_2 satisfies the inequality for any $\delta > 0$ for $\epsilon = \frac{1}{2}$. Hence, proved.
2. $\phi(0^+) = \frac{x^a sin(\frac{1}{x^c})-0}{x-0} = x^{a-1}sin(\frac{1}{x^c})$. Similarly, $\phi(0^-) = -(x^{a-1}sin(\frac{1}{x^c}))$. For the limit to exist, the left-hand limit and the right-hand limits must exist and be equal. This, in turn, is possible if and only if $a > 1$ (from above).
3. This is clear from the form of the derivative, $f'(x) = sgn(x)[a|x|^{a-1}sin(|x|^{-c}) - cx^{a-1-c}cos(|x|^{-c})]$.
4. At $a = 1 + c$, the derivative will have a discontinuity of the second kind at $x = 0$, while at $a > 1 + c$, it tends to 0 as x tends to 0 and is thus continuous. (Note that $a < 1 + c$ is not possible since it must be bounded.)

   The remaining parts are virtually identical.

Answer 14.

1. Suppose f is convex on (a, b). Then, $\forall x_1, x_2 \in (a, b), f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) \forall t \in [0, 1]$. Let $t = \frac{1}{2}, x_1 = x, x_2 = x + 2h$. Then, $2f(x+h) \leq f(x) + f(x+2h) \implies f(x+h) - f(x) \leq f(x+2h) - f(x+h)$. If we set $x_1 = x+h, x_2 = x+3h$, then we similarly get $f(x+2h) - f(x+h) \leq f(x+3h) - f(x+2h)$.
   Setting $h = \frac{x_2 - x_1}{n}$ and iterating this n times, we get $f(x_1 + h) - f(x_1) \leq f(x_2 + h) - f(x_2)$. Dividing both sides by h and letting $n \to \infty$, we get $f'(x_1) \leq f'(x_2)$. Thus, $f'(x)$ is monotonically increasing on (a, b).
   Next, suppose f' is monotonically increasing on (a, b). Then, $f'(d) \geq f'(c), c \in (x, z), d \in (z, y)$. Using the mean value theorem, we have $\frac{f(y)-f(z)}{y-z} \geq \frac{f(z)-f(x)}{z-x}$.
   Substituting $t = \frac{y-z}{y-x}$ into this gives us $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$.
2. The corresponding result for $f''(x)$ follows from the fact that if $f''(x)$ exists, then $f'(x)$ is monotonically increasing if and only if $f''(x) > 0$.

Answer 15. By Taylor's theorem, for $\beta = x+2h, \alpha = x$, we have $f(x+2h) = f(x) + 2hf'(x) + 2h^2 f''(c)$, with $a < x < c < x + 2h$.
Rearranging, we get $f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(c) \implies f'(x) \leq \frac{M_0}{h} + hM_2 \implies hM_1 \leq M_0 + h^2 M_2 \implies h^2 M_2 - hM_1 + M_0 \geq 0$.
For this quadratic in h to be nonnegative everywhere, $D \leq 0 \implies M_1^2 \leq 4M_0M_2$. Hence, proved.

Answer 16. From above, we have $f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(c)$. Taking the limit $x \to \infty$ on both sides, we get $lim_{x\to\infty} f'(x) = -hf''(c)$. Taking the limit $h \to 0$ on both sides, we now get $lim_{h\to 0} lim_{x\to\infty} f'(x) = lim_{x\to\infty} f'(x) = 0$. Hence, proved.

Answer 17. By Taylor's theorem, for $\beta = 1, \alpha = 0$, we have $f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f'''(c)}{6} \implies 1 = \frac{f''(0)}{2} + \frac{f'''(c)}{6}$ for some $c \in (0,1)$. Similarly, for $\beta = -1, \alpha = 0$, we have $f(-1) = -f(0) - f'(0) + \frac{f''(0)}{2} + \frac{f'''(d)}{6} \implies 0 = \frac{f''(0)}{2} - \frac{f'''(d)}{6}$ for some $d \in (-1,0)$. Subtracting the two equations, we get $f'''(c) + f'''(d) = 6 \implies f'''(x) > 3$ for some $x \in (-1,1)$.