Basic Topology

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Answer 1. We wish to prove that $x \in \emptyset \implies x \in A$ for any set A. Since the left-hand side of the implication is always false, the implication itself is true; therefore, $\emptyset \subset A$ for any set A.

Answer 2. From theorem 2.12 and theorem 2.13, we can conclude that the set of all n-tuples of integers is enumerable. Since every algebraic number is associated with at least one n-tuple, it follows that the set of algebraic numbers is enumerable.

Answer 3. If every real number was algebraic, since the set of all algebraic numbers is enumerable, this would mean that the set of real numbers is enumerable, which is a contradiction. There must be non-enumerably many real numbers which are not algebraic.

Answer 4. We know that $\mathbb{R}$ is uncountable, and that $\mathbb{Q}$ is countable, and that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, where $\mathbb{Q}^c$ is the set of irrational real numbers. Since the union of two countable sets must also be countable, we conclude that $\mathbb{Q}^c$ is not countable.

Answer 5. Let $A_k = \bigcup\{k + \frac{1}{n}, n \in \mathbb{N}\}$. $A_1 \cup A_2 \cup A_3$ is a bounded set with limit points $\{0, 1, 2\}$.

Answer 6. If x is a limit point of E', every neighbourhood N_x around it contains another point in E', i.e., every neighbourhood around it contains at least one limit point $q \neq x$ of E, that is, such that every neighbourhood N_q around q contains at least one point $p \neq q, p \in E$.
For any given neighbourhood N_x, let us choose a neighbourhood N_q around the point of concern q in it such that $N_q \subset N_x$. This ensures that some $p \in N_x$ for every neighbourhood; in other words, x is also a limit point of E. E' contains all its limit points and is thus closed.
Since $E \subset \bar{E}$, it is trivial that a limit point of E will be a limit point of $\bar{E}$. Suppose x is a limit point of $\bar{E}$. This means that every neighbourhood N_x around it contains another point q. If $q \in E'$, we can find a point $p \in E$ such that $p \in N_x$, as above. If $q \in E$, then we have already found a point in the neighbourhood which is in E.
Thus, if x is a limit point of $\bar{E}$, then every neighbourhood around it will contain some point in E; it follows that it is also a limit point of E.
While every limit point of E' is a limit point of E, the converse need not hold. Consider $\bigcup\{\frac{1}{n}, n \in \mathbb{N}\}$. Here, the only limit point of E is 0, but this is not a limit point of the singleton E'.

Answer 7.

1. Suppose $x \in \bar{B_n}$ such that it is a limit point of B_n which is not in B_n. This means that, in every neighbourhood around it, there is a point q such that $q \in A_i$ for some i. In other words, it is a

limit point of $\bigcup_{i=1}^{n} A_i$. Therefore, $x \in \overline{\bigcup_{i=1}^{n} A_i} \implies x \in \bigcup_{i=1}^{n} \bar{A}_n$ (since $\overline{A \cup B} = \bar{A} \cup \bar{B}$).
Since $A \subseteq \bar{A}$ in general, it is immediate that $x \in \ B_n \implies x_n \in \bigcup_{i=1}^{n} \bar{A}_i$. Thus, $\bar{B_n} \subset \bigcup_{n=1}^{\infty} \bar{A}_i$.
Next, suppose $x \in \bigcup_{i=1}^{n} \bar{A}_i \implies x \in \bar{A}_i$ for some i. Suppose x is a limit point of A_i which is not in A_i. This means that every neighbourhood of x contains a point of A_i, and so, every neighbourhood of it contains a point of B_n (as elaborated below). Thus, x is a limit point of B_n, and $x \in \bar{B_n}$.
If $x \in A_i$, then $x \in B_n \implies x \in \bar{B_n}$. Thus, $\bigcup_{i=1}^{n} \bar{A}_i \subset \bar{B_n}$, and this completes the proof.

2. Suppose $x \in \bigcup_{i=1}^{\infty} \bar{A}_i$. This means that either $x \in A_i$ or x is a limit point of A_i for some i. If the former, we have $x \in \bar{B}$. If the latter, then for every neighbourhood around it, there exists a point $q \in A_i \implies q \in B$. Therefore, x is a limit point of B as well, implying $x \in \bar{B}$, and this completes the proof.
$A_i = \frac{1}{i}$ would be an example wherein the inclusion would be proper, because 0 would only be in $\bar{B}$.

Answer 8. Yes and no. Suppose $x \in E$ and it is not a limit point. This means that there exists a neighbourhood N_x such that $N_x \cap E = \emptyset$ or x. In any case, N_x will not be a subset of E, and that would mean x is not an interior point of E, contradicting the fact that E is an open set.
Consider the closed set $[0, 1] * [0, 1] \cup (2, 2)$. $(2, 2)$ is an element in it but not a limit point.

Answer 9.

1. We need to prove that every point of E^0 is an interior point of it; that is, an interior point of E is an interior point of E^0. Consider the neighbourhood N_ϵ around the interior point $x \in E$ such that $N \subset E$, and pick $q \in N$ such that $d(q, x) = \delta$. Now, pick a neighbourhood $N_{\frac{\epsilon-\delta}{2}}$ around q. Clearly, this new neighbourhood is a subset of N_ϵ; therefore, q is also an interior point for every $q \in N_\epsilon$. From this we have $q \in E^0 \implies x$ is an interior point of E^0, completing the proof.

2. If $E = E^0$, it follows from the fact that E^0 is open. If not, there is a point $x \in E$ which is not an interior point; and so, by definition, E is not open.

3. Since G is open, there exist a neighbourhood N around $x \in G$ such that $N \subset G$. But since $G \subset E$, we have $N \subset E$; so that x is an interior point of E as well. Thus, $x \in G \implies x \in E^0$, and the proof is complete.

4. $x \in E^0c \implies x \in E^c$ or $x \in E$ such that it is not an interior point. If the former, it follows that it is in the closure of E^c. If the latter, then every neighbourhood around x is such that it contains at least one point of E^c, making x a limit point of E^c and an element of its closure. Hence, proved.

5. No. Let $E = \mathbb{Q}$. Due to the Archimedean property, this has no interior points, and $E^0 = \emptyset$. However, $\bar{E} = \mathbb{R}$, for which every real number is an interior point.

6. No. Once again, let $E = \mathbb{Q}$. Since $E^0 = \emptyset$, its closure remains $\emptyset$. However, $\bar{E} = \mathbb{R}$.

Answer 10.

1. $d(p, q) = 1 > 0, p \neq q$. Also, $d(p, q) = 0, p \neq q$.

2. If $p = q, d(p, q) = d(q, p) = 0$. Otherwise, $d(p, q) = d(q, p) = 1$.

3. If $p \neq q \neq r, 1 = d(p, q) < d(p, r) + d(r, q) = 2$. If $p = q \neq r, 0 = d(p, q) < d(p, r) + d(r, q) = 2$. If $p \neq q = r$, then $1 = d(p, q) = d(p, r) + d(r, q) = 1 + 0 = 1$.
Thus, $d(p, q) \leq d(p, r) + d(r, q), r \in X$.

Therefore, d is a metric.

1. Every subset will be open. Pick an arbitrary subset E with $x \in E$. Consider a neighbourhood N with radius, say, 0.5. The only element of N will be x, and as such, $N \subset E$, making x an interior point. This holds for all the points in E, and so E is open.
2. Every subset will be closed, since none will have any limit points. (For any point $x \in E$, if we pick a neighbourhood of radius less than 1, it will contain only x, violating the definition of a limit point.)
3. Only sets with finitely many elements will be compact, since infinite sets can have an open cover consisting of each of their elements (since even a point is an open set) which will afford of no finite subcover.

Answer 11.

1. No. For $x = -1, y = 0, z = 1, 4 = d(x, z) > d(x, y) + d(y, z) = 2$.
2. Yes. It is clear that the first two conditions are satisfied. We also have $\sqrt{|x-z|} \leq \sqrt{|x-y|} + \sqrt{|y-z|}$ from the triangle inequality, which one can see by squaring both the sides.
3. Since $d(1, -1) = 0$ and $1 \neq -1$, this cannot be a metric.
4. Again, since $d(1, \frac{1}{2}) = 0$ and $1 \neq \frac{1}{2}$, this cannot be a metric.
5. Yes. It is again clear that the first two conditions are satisfied. For the third, let $a = |x-z|, b = |x-y|, c = |y-z|$. We know that $a \leq b+c$. We need to show that $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. Simplifying this yields $a \leq b + c + 2bc + abc$, which is true since $a, b, c \geq 0$.

Answer 12. Let G_α be an open cover of K. Specifically, let G_0 be the open set such that $0 \in G_0$. Since G_0 is open, 0 must be an interior point, and there will be some neighbourhood around 0 such that $N_\epsilon \subseteq G_0$. This means that $\frac{1}{n} \in G_0$ for all $n > \frac{1}{\epsilon}$. The numbers less than $\frac{1}{\epsilon}$ will obviously be finite; say that they form the set $\{1, 2...n_\epsilon\}$. Each of these will be associated with at least one set in the open cover G_n. Therefore, the collection $\bigcup_{k=0}^{n} G_k$ will be a finite subcover.

Answer 13. We claim that $K = \{0\} \cup \{\frac{1}{n} : n = 1, 2...\} \cup \{\frac{1}{m} + \frac{1}{n} : n = m, m+1..., m = 1, 2...\}$ is the required set.
It is clear that $\{0\} \cup \{\frac{1}{m} : m = 1, 2...\}$ are the (countably many) limit points (all of which are contained in the set), and that the set is bounded below by 0 and above by 2. Thus, by the Heine-Borel theorem, it is also compact.
To see that the set cannot have any *other* limit points, pick an $x \neq \frac{1}{m}, x \in (0, 1)$ such that $\frac{1}{p+1} < x < \frac{1}{p}$. Choose a neighbourhood N_ϵ with $\epsilon = \frac{1}{2} min(x - \frac{1}{p+1}, \frac{1}{p} - x)$. $K \cap (x - \epsilon, x + \epsilon) \subseteq \{\frac{1}{1+p} + \frac{1}{k} : k < \frac{1}{\epsilon}\} \cup$. Since the RHS is a finite set, x cannot be a limit point.

Answer 14. Consider the open cover $\bigcup_{k=1}^{\infty} G_k$, where $G_k = (0, \frac{k-1}{k})$. Any finite intersection (say, upto $k = m$) will fail to contain the points in $(\frac{1}{m}, 1)$.

Answer 15. The collection of closed sets $B_n = [n, \infty)$ satisfy the conditional but have an empty intersection. Similarly, the collection of bounded sets $A_n = (0, \frac{1}{n})$ satisfy the conditional but have an empty intersection.

Answer 16. It is clear that the set has the lower and upper bounds $-3, 3$ respectively; thus, it is bounded.
Let p be a limit point of E, and let N_ϵ be a neighbourhood around it containing $q \in E, d(p,q) < \epsilon, \epsilon \in \mathbb{Q}$. Without loss of generality, suppose $q < p$. Then, it is clear that $2 < q^2 < p^2$.

Let U_n be the collection of sets defined as $\{p : 2 < p^2 < 3 - \frac{1}{n}\}$. By the same argument to be used to show E is open, U_n forms an open cover of E. But clearly, it can have no finite subcover; thus, E is not compact.
E is open in $\mathbb{Q}$. For any point $p \in E$, we can pick a neighbourhood N_ϵ around it such that $\epsilon <$min (), ensuring that $(p-\epsilon, p+\epsilon) \in E$.

Answer 17. E is not countable. We can use a Cantor-esque argument and change the nth digit of the nth number in the enumeration from 4 to 7 or vice-versa to find something not in the enumeration.
E is not dense in [0,1]. The 0.1-radius neighbourhood around 0.9 contains no point in E, and so we have a point in [0,1] which is not a limit point of E nor in E.
It is clear that E is bounded. To see that it is closed, take any limit point p of E and suppose $p \notin E$. Thus, E is compact.

Answer 18.

Answer 19.