The Real and Complex Number Systems

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June 2021

Answer 1. Suppose $r + x = \frac{p}{q}$ for some integral p, q and rational r. Then, $x = \frac{p-qr}{r}$, contradicting its irrationality. Similarly, if $rx = \frac{p}{q}$ for some integral p, q, then $x = \frac{p}{qr}$, contradicting its irrationality.

Answer 2. $x = \sqrt{12} = 2\sqrt{3}$. Since 2 is rational and $\sqrt{3}$ is irrational, x is irrational (from the previous problem).

Answer 3.

1. Multiply both sides by $\frac{1}{x}$. Axioms 3, 4 and 5 are used here.
2. Apply the above with $z = 1$.
3. Apply the above with $z = \frac{1}{x}$.
4. Axiom 5 on $\frac{1}{x}$.

Answer 4. By the definition of upper and lower bounds, we have $x \in E$ such that $\alpha \le x \le \beta$. By the transitivity of ordering, we have $\alpha \le \beta$.

Answer 5. Let $\inf A = l$. We have to show that $\sup -A = -l$. Note that $x \ge y \implies x - x - y \ge y - x - y \implies -y \ge -x$.
Therefore, $l \le x \forall x \in A \implies -l \ge -x \forall x \in A$. This shows that $-l$ is an upper bound of $-A$.
Suppose $-a < -l$ is the supremum of $-A$ (and so, that $-a \ge -x \forall x \in A$). This means that $x > a > l \forall x \in A$, contradicting the fact that $\inf A = l$. Hence, proved.

Answer 6.

1. We use the fact that $(b^x)^y = b^{xy}$ for rationals x, y.
   $((b^m)^{\frac{1}{n}})^{nq} = b^{mq}$. But also, $((b^p)^{\frac{1}{q}})^{nq} = b^{np} = b^{mq}$. By uniqueness of nth root, we have the desired equality.
2. Let $r = \frac{p}{q}, s = \frac{m}{n}$. Then $r + s = \frac{np+mq}{nq}$. Therefore, $b^{r+s} = (b^{np+mq})^{\frac{1}{nq}} = b^{\frac{p}{q}} b^{\frac{m}{n}} = b^{rs}$. Hence, proved.
3. First, we prove that b^r is an upper bound. It suffices to show that $\alpha < \beta \implies x^\alpha < x^\beta, x > 1, \alpha, \beta \in \mathbb{Q}$.
   Suppose instead that $x^\alpha > x^\beta$. Let n be the product of the denominators of α, β. Then $x^{n(\alpha-\beta)} \ge 1$. However, $n(\alpha - \beta)$ is a negative integer. This is a contradiction.
   Next, suppose that there exists an upper bound x such that $x < b^r$. But since r is rational, $b^r \in B(r)$, and so x cannot be an upper bound.

4. Note that b^{x+y} =sup $B(x+y)$, where $B(x+y)$ is the set of all numbers $b^t, t < x+y$. Further, we know that $b^{r+s} = b^r b^s$ for rationals r, s.
   We have to show that sup $B(x)*$sup $B(y)$ =sup $B(x+y)$.
   First, we show that $B(x+y) \leq$ sup $B(x)*$sup $B(y)$. Suppose $r < x, s < y \implies r+s < x+y$. $b^r b^s = b^{r+s} <$ sup $B(x)*$sup $B(y), b^r \in B(x), b^s \in B(y), b^{r+s} \in B(x+y)$. Thus, sup $B(x)*$sup $B(y)$ is an upper bound for $B(x+y)$.
   Furthermore, $a <$sup $B(x)*$sup $B(y) \implies \frac{a}{supB(x)} < m < supB(y)$, where $m = \frac{1}{2}(\frac{a}{supB(x)} + supB(y))$. Thus, there exists $b^r \in B(x), b^s \in B(y)$ such that $\frac{a}{m} < b^r, m < b^s \implies (\frac{a}{m})m < b^r b^s = b^{r+s} \in B(x+y) \implies a$ cannot be an upper bound to $B(x+y)$.
   Hence, proved.

Answer 7.

1. We prove this by induction. For $n = 1$, the equation reads $b - 1 \geq b - 1$, which is true. Now, suppose it is true for some n. $b^{n+1} - 1 = b(b^n - 1) + (b - 1) \geq n(b - 1) + (b - 1)$ (by induction hypothesis) $= (n+1)(b-1)$. Hence, proved.

2. Setting $b^n = a$ yields $a - 1 \geq a^{\frac{1}{n}} - 1$.

3. We have $n \geq \frac{b-1}{t-1}$. But also, $\frac{b-1}{b^{\frac{1}{n}}-1} \geq n$. Thus, $\frac{b-1}{b^{\frac{1}{n}}-1} > \frac{b-1}{t-1} \implies b^{\frac{1}{n}} < t$.

4. Setting $t = \frac{y}{b^w}$, an application of the above yields, for sufficiently large n and $y > b^w$, $b^{w+\frac{1}{n}} < y$.

5. Setting $t = \frac{b^w}{y}$, an application of the above yields, for sufficiently large n and $y < b^w$, $b^{w-\frac{1}{n}} > y$.

6. Suppose $b^x < y$. This means that, by 4, $b^{x+\frac{1}{n}} < y$ for large n, contradicting the fact that x is an upper bound. Suppose $b^x > y$. Then, by 5, $b^{x-\frac{1}{n}} > y$ for large n, contradicting the fact that x is the smallest upper bound. Thus, $b^x = y$.

7. Suppose $z > x$. Then, $b^z = b^{z+x-x} = b^x b^{z-x} > b^x$. Similarly, $b^z < y$ for $z < x$. Thus, x is unique.

Answer 8. We know that, in any ordered field, $x^2 > 0\ \forall x$. Setting $x = i$, we have $-1 > 0$. But also, $x > 0 \implies -x < 0$. Thus, $1 = (1)^2 < 0$, a contradiction.

Answer 9.

1. We have $z, w \in \mathbb{C}$. If $a = c, b = d$, then $z = w$. Else, if $a = c, b < d$ or $b > d$, then $z < w$ or $z > w$ respectively. Else, if $a < c$ or $a > c$, then $z < w$ or $z > w$ respectively. Since we know that $\mathbb{R}$ is an ordered set, this covers all the possibilities. Therefore, we have either $z = w, z > w$ or $z < w$.

2. This follows similarly from the order of $\mathbb{R}$.

Answer 10. $z^2 = a^2 - b^2 + 2abi = u + \sqrt{|w|^2 - u^2}i = u + \sqrt{u^2 + v^2 - u^2}i = u + i|v| = u + iv = w$ for $v \geq 0$.
$\bar{z}^2 = a^2 - b^2 - 2abi = u - \sqrt{|w|^2 - u^2}i = u - i|v| = u - i(-v) = u + iv = w$ for $v \leq 0$.

Answer 11. Let $z = x + iy, w = a + ib, a^2 + b^2 = 1$. We want $ra = x, rb = y \implies \frac{x}{a} = \frac{y}{b}$. Setting $a = \frac{x}{\sqrt{x^2+y^2}}, b = \frac{y}{\sqrt{x^2+y^2}}$ gives us this.

Answer 12. We prove the given by induction.
It is known that $|z_1 + z_2| \leq |z_1| + |z_2|$. Suppose now that it is true that $|z_1 + ...z_n| \leq |z_1| + ...|z_n|$.
$|z_1 + ...z_n + z_{n+1}| \leq |z_1 + ...z_n| + |z_{n+1}|$ ($n = 2$ case) $\leq |z_1| + ...|z_n| + |z_{n+1}|$ (by induction hypothesis).

Hence, proved.

Answer 13. Let $x = y + z$ for some z. The triangle inequality gives us $|z+y| \leq |z| + |y| \implies |z+y| - |y| \leq |z| \implies |x| - |y| \leq |x - y|$. If $|x| < |y|$, this is automatically true. If $|x| > |y|, |x| - |y| = ||x| - |y||$. Therefore, $||x| - |y|| \leq |x - y|$.

Answer 14. 4

Answer 15. The equality holds either if (following the notation of theorem 1.35) $B = 0$ or $a_j = \frac{C}{B} b_j$ for all j, that is, **a, b** are linearly dependent.

Answer 16.

1. Set $|\mathbf{w}| = \sqrt{r^2 - \frac{d^2}{4}}, \mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) = 0, \mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2} + \mathbf{w}$. It is clear that there are infinitely many $\mathbf{z}$, and it is easy to check that any of them will satisfy the given equation.
2. Suppose there exists such a $\mathbf{z}$. Then, $d = |\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}| \leq |\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{y}| = 2r$ (by assumption). However, it is given that $2r = d$. Thus, $|\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}| = |\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{y}|$. We know from 15 that an equality will only hold if the two vectors are linearly dependent, that is, $\mathbf{x} - \mathbf{z} = c(\mathbf{z} - \mathbf{y}) \implies \mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}$.
3. Suppose there exists such a $\mathbf{z}$. Then, $d = |\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z} + \mathbf{z} - \mathbf{y}| \leq |\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{y}| = 2r$ (by assumption). However, it is given that $2r < d$. This is a contradiction. Thus, no such $\mathbf{z}$ exists.

For $k = 2$, there are precisely two solutions in case 1. For $k = 1$, there are no solutions in case 1. The remaining two cases are unchanged.

Answer 17. $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{x} - 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$.
If we take **x, y** to be the sides of a parallelogram, this says that the sum of the squares of the side lengths equals the sum of the squares of the diagonal lengths.

Answer 18. Set $y_i = \frac{(-1)^i}{x_i}$; if $x_i = 0, y_i = 1$. If k is odd, then $y_k = 0$.
This is not true for $k = 1$.

Answer 19. Using the solutions given, squaring and simplifying the two equations shows that they are equivalent.

Answer 20.

1. Let $A \subset R$ be a non-empty set bounded above, and denote by γ the union of all the elements of A. γ is non-empty and, since A is bounded above, $\gamma \neq Q$. Furthermore, if $p \in \gamma$, $q < p, q \in Q$, then $q \in A$ (since γ is an upper bound), and so by definition, $q \in \gamma$. Thus, $\gamma \in R$.
   Suppose $\delta < \gamma$. Then $r \in \gamma, r \notin \delta$ for some r. But $r \in \gamma \implies r \in A$, and so δ is not an upper bound. Thus, γ is a supremum.
2. Let $r \in \alpha$. Let 0^* be the cut with largest number 0. $r = r + 0$. Thus, $r \in \alpha + 0^*, \alpha \subset \alpha + 0^*$. For the other way round, notice that $r + s \leq r$ for $r \in \alpha, s \in 0^*$. Thus, $\alpha + 0^* \subset \alpha$.
3. No set of the form $S = \{r : r < q\}$ will have an additive inverse, since it has no largest element (and thus, nor will it sum with any other cut), whereas 0^* does.

The proof for properties A1-A3 is virtually unchanged from what was given in the text.

Postscript.

Theorem: Every complete ordered field is isomorphic to R.

Proof. Let F be a complete ordered field with operations $\bigoplus$ and $\bigodot$. Define the bijection $f : R \to F$ on integers as follows:

1. $f(0) = 0_F$
2. $f(n) = (1_F \bigoplus 1_F \bigoplus 1_F ... \bigoplus 1_F)$ n times when $n > 0$
3. $f(n) = -(1_F \bigoplus 1_F \bigoplus 1_F ... \bigoplus 1_F)$ n times when $n < 0$

Define f on rational numbers by $f(\frac{m}{n}) = m_F \bigodot n_F^{-1}$. (Note that f is now an order-preserving bijection from Q to Q_F.)
Finally, let us define a set $S(r) = \{f(q) : q < r, q \in Q\}$ for any arbitrary real number r, and set $f(r) = \sup S(r)$. We claim that f is the required isomorphism.

1. $f(x)$ is well-defined: It is clear that the function is well-defined on all rationals. For the rest, we only need to show that $S(r)$ is non-empty and bounded above; it follows from the assumption of completeness that a supremum exists.
   It is non-empty because, by the Archimedean property, there exists an integer $n > |r| \implies -n < r \implies n_F \in S(r)$. Since n_F will also be an upper bound on $S(r)$, it is also bounded above.
2. $f(x)$ is a bijection:
3. $f(x + y) = f(x) \bigoplus f(y)$:
4. $f(x * y) = f(x) \bigodot f(y)$:

Hence, proved.