Cohomology of graded Lie algebras

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Motivation

You can do it!

Kidding. In the award letter, it is said that the motivation for this project 'comes from rational homotopy theory. Those cohomology groups appear in the classification of rational homotopy types having prescribed homotopy groups endowed with Whitehead products.' Let us try to understand what all this is about.

Definition. *Let X be a simply connected topological space. Then, its rational homotopy groups are the groups $\boldsymbol{\pi}_{n+1}(X) := \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Definition (Whitehead product)**.** *Let $f \in \pi_l(X), g \in \pi_k(X)$. Furthermore, let $\phi : S^{k-l-1} \to S^k \vee S^l$ be the attaching map.*
Then, the ***Whitehead product*** $[f, g] := (f \vee g) \circ \phi \in \pi_{k+l-1}(X)$.

When equipped with the Whitehead product, the groups $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ form a graded Lie algebra; this, we denote by $\boldsymbol{\pi} X$.

The following results, due to Quillen, are what tie our project into the broader endeavour of rational homotopy theory:

Theorem 1.1 (Quillen)**.** *Given a simply connected space X, there exists a differential graded Lie algebra L such that $H_*(L) \cong \boldsymbol{\pi} X$.*

Corollary 1.1.1. *Given a reduced graded Lie algebra L, there exists a simply connected space X such that $L \cong \boldsymbol{\pi} X$.*

Category theory

Discussion. We first define some important classes of functors.

- **Adjoint functors:** Let $\mathbb{A} \xrightarrow{F} \mathbb{B}, \mathbb{B} \xrightarrow{G} \mathbb{A}$ be categories and functors. We say that F is *left adjoint* to G (G is *right adjoint* to F) if, for $A \in \mathbb{A}, B \in \mathbb{B}$, there is a *natural isomorphism* $\mathrm{Hom}(F(A), B) \to \mathrm{Hom}(A, G(B))$.

  Let us denote the bijection by ψ, and write $\psi(f) = \overline{f}$. Naturality means that, for each A, B, ψ satisfies the following two conditions:

  1. For $g : F(A) \to B, q : B \to B', \overline{q \circ g} = G(q) \circ \overline{g}$.
  2. For $f : A \to G(B), p : A' \to A, \overline{f \circ p} = \overline{f} \circ F(p)$.

  For the remainder, we shall work in the category of R-modules, where R is a commutative ring.

- **Right exact functors:** Let $0 \to A \to B \to C \to 0$ be a short exact sequence of $R-$modules. Then, a functor $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is said to be *right exact* if $F(A) \to F(B) \to F(C) \to 0$ is an exact sequence.
  An example is provided by the tensor product $F := M \otimes_R —$, where M is an $R-$module. If this functor is also left exact, i.e., it is exact, we say M is *flat*.

- **Left derived functors:** Given a right exact functor F, its left derived functors are the functors $L^i F, i \geq 1$, such that the sequence $... \to L^2F(C) \to L^1F(A) \to L^1F(B) \to L^1F(C) \to A \to B \to C \to 0$ is exact.
  The left derived functors of the tensor product are the Tor functors.

Left exact functors and right derived functors are the obvious extensions of the above definition. An example of a left exact functor is $\mathrm{Hom}_R(M, -)$. If this is also right exact, i.e., it is exact, we say M is *projective*. The right derived functors of Hom are the Ext functors.

Remark. The existence of derived functors is contingent on the category satisfying certain conditions. That of R-modules happens to satisfy them.

Definition (Projective resolution)**.** *Let R be a commutative ring and M be a module. Then, a* ***projective resolution*** *of M is an exact sequence*

$$... \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

such that each P_i is a projective R-module.

Remark. One can check that the following construction of left derived functors sastisfies our original definition.

Definition (Left derived functors)**.** *Let $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a right exact functor. Given a module M, let $P \to M$ be a projective resolution. We may define, for $i \geq 0$,*

$$L_i F(M) := H_i(F(P))$$

where we take the homology after cutting off $F(M)$.

Remark. One needs to briefly check that $F(P)$ is a chain complex before defining homology groups. Let us do this explicitly. The sequence at hand is as follows:

$$... \to F(P_2) \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \to 0$$

Now, $F(d_i) \circ F(d_{i+1}) = F(d_i \circ d_{i+1}) = F(0)$ (exactness of projective resolution) $= 0$. Thus, we have a chain complex on our hands, and taking homology makes sense.
Also, note that $F(P_0)/\mathrm{Im}(F(d_1)) \cong F(P_0)/\ker(F(\epsilon))$ (right exactness) $\cong F(M)$ (right exactness; first isomorphism theorem). Therefore, $H_0(F(P)) = L_0(F(M)) \cong F(M)$.

Lemma 2.2. *Every R-module M has a projective resolution.*

Lemma 2.3. *If $P \to M, Q \to M$ are two projective resolutions, then $H_i(F(P)) \cong H_i(F(Q))$.*

Remark. Lemma 1.1 ensures existence of the left derived functors, and lemma 1.2 ensures uniqueness.

Example. Let us first clarify notation. When we write $\mathrm{Tor}_n^R(A, B)$, we mean the n^{th} Tor group of B for the right exact functor $A \otimes_R —$. We will now compute the Tor groups of an arbitrary abelian group ($\mathbb{Z}$-module) B corresponding to $\mathbb{Z}_p$.
As such, we would have to begin with a projective resolution for B. However, it turns out that $\mathrm{Tor}_n^R(A, B) = \mathrm{Tor}_n^R(B, A)$ (although this is non-trivial!). (To put it another way, $L_n(A \otimes_R -)(B) \cong L_n(- \otimes_R B)(A)$.) Therefore, we can work with functor $B \otimes_{\mathbb{Z}} —$ and module $\mathbb{Z}_p$ instead.
A projective resolution of $\mathbb{Z}_p$ is the following:

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$$

(Recall that $\mathbb{Z}$ is projective.)
Acting with the relevant functor, using $R \otimes M \cong M$ and discarding the last term, we get the following chain complex:

$$0 \to B \xrightarrow{p} B \to 0$$

Completing the computation of the homology groups, we finally have
$\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}_p, B) = B/pB (= \mathbb{Z}_p \otimes B), \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, B) = \{b \in B : pb = 0\}, \mathrm{Tor}_n^{\mathbb{Z}}(\mathbb{Z}_p, B) = 0, n \geq 2.$

Lemma 2.4. *Projective modules are flat.*

Proof. Let M be a projective $R-$module. If we can show that $\mathrm{Tor}_n^R(M, A) = 0$ for all $n \geq 1$ and $R-$modules A, flatness follows definitionally.
But since M is projective, its projective resolution is simply $0 \to M \xrightarrow{id} M \to 0$. Upon applying the functor and discarding the end, we simply have $0 \to M \otimes A \to 0$, which has trivial homology groups of order > 0. Hence, proved. □

Remark. More generally, if A is a projective module, $L_i F(A) = 0, i > 0$.

Discussion. Our understanding of projective modules was based on the left-exactness of the functor $\mathrm{Hom}_R(M, -)$. As it so happens, the functor $\mathrm{Hom}_R(-, M)$ is also left exact. However, a different class of modules make it exact. If $\mathrm{Hom}_R(-, M)$ is exact for some module M, we say M is *injective.*

Definition (Injective resolution)**.** *Let R be a commutative ring and M be a module. Then, an **injective resolution** of M is an exact sequence*

$$0 \to M \xrightarrow{\epsilon} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \ldots$$

such that each I_i is an injective R-module.

Definition (Right derived functors)**.** *Let $F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ be a left exact functor. Given a module M, let $M \to I$ be an injective resolution. We may define, for $i \geq 0$,*

$$R^i F(M) := H^i(F(I))$$

where we take the cohomology after cutting off $F(M)$ from the cochain complex.

Remark. The following results will carry forward from the discussion on left derived functors:

- Right derived functors exist and are unique.
- $R^0(F(M)) \cong F(M)$.
- If I is an injective module, $R^i(F(I)) = 0, i > 0$.

Example. Let us compute $\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}_p, B)$, where B is an arbitrary abelian group. The notation used emulates that of Tor.
We can begin with a projective resolution of $\mathbb{Z}_p$ (see remark below for clarification), given by the following:

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}_p \to 0$$

Next, we act on this with the functor $\mathrm{Hom}(-, B)$. Using the fact that $\mathrm{Hom}(\mathbb{Z}, B) \cong B$, the following cochain complex is the relevant one:

$$0 \leftarrow B \xleftarrow{p} B \leftarrow 0$$

Completing the computation of the cohomology groups, we finally have
$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_p, B) = \{b \in B : pb = 0\} = \mathrm{Hom}(\mathbb{Z}_p, B), \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p, B) = B/pB, \mathrm{Ext}^{\mathbb{Z}}_n(\mathbb{Z}_p, B) = 0, n \geq 2.$

Remark. Instead of starting with an injective resolution for B, we have worked with a projective resolution for $\mathbb{Z}_p$. The following facts validate this switch:

- $R^*\mathrm{Hom}(A, -)(B) \cong R^*\mathrm{Hom}(-, B)(A)$
- $\mathrm{Hom}(-, B)$ is a contravariant functor
- Injective resolutions in the opposite category are equivalent to projective resolutions in the original category.

Theorem 2.5. *If F is left adjoint to G, then F is right exact and G is left exact.*

Interlude: Algebras

Definition (Graded Algebra). *A graded ring R is a ring that can be decomposed into a direct sum $\bigoplus_{n=0}^{\infty} R_n$ of additive groups such that $R_n R_m \subseteq R_{m+n}$.*
An algebra A over a ring R is a graded algebra if it is graded as a ring.

Definition (Filtered Algebra). *A filtered algebra A over a ring R is an algebra such that there is an increasing sequence of subspaces $\{0\} \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq A$ which satisfy $F_m \cdot F_n \subseteq F_{m+n}, \bigcup_{i \in \mathbb{N}} F_i = A$.*

Remark. A filtered algebra is a generalization of a graded algebra, since every graded algebra is filtered by setting $F_n = \bigoplus_{i=0}^{n} R_i$. Note that the filtration could be an infinite sequence on *either* side.

Definition (Associated Graded Algebra). *Let A be a filtered algebra. Then, define $\mathfrak{G}(A) = \bigoplus_{n=0}^{\infty} G_n$, where $G_0 = F_0, G_n = F_n / F_{n-1}$.*
When endowed with the naturally induced multiplication map, this turns into a graded algebra, called the associated graded algebra of A.

Lie algebras

Definition (Lie algebra)**.** *Let R be a commutative ring. A* ***Lie algebra*** $\mathfrak{g}$ *is an R-module equipped with a bilinear product, $(x, y) \mapsto [xy]$, which satisfies skew-symmetry and the Jacobi identity.*

Definition (Ideal)**.** *An* ***ideal*** *of a Lie algebra $\mathfrak{g}$ is a submodule $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$.*

Remark. A Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a linear map such that $\phi([x, y]) = [\phi(x), \phi(y)])$.

Definition ($\mathfrak{g}$-modules)**.** *A (left) $\mathfrak{g}$-module M is an R-module with an R-bilinear product $\mathfrak{g} \times M \to M, (x, m) \mapsto xm$ such that $[x, y]m = x(ym) - y(xm)$.*

Remark. A $\mathfrak{g}-$module homomorphism $f : M \to N$ is an R-module homomorphism which also satisfies $f(xm) = xf(m)$. We denote the set of all $\mathfrak{g}-$module homomorphisms by $\mathrm{Hom}_{\mathfrak{g}}(M, N)$. This is an $R-$submodule of $\mathrm{Hom}_R(M, N)$.

Definition (Tensor algebra)**.** *Let M be an R-module. Then, the* ***tensor algebra*** *$T(M)$ is the following associative algebra:*
$T(M) = R \oplus M \oplus (M \otimes M) \oplus ... \oplus M^{\otimes n} \oplus ...$

Remark. Here, $M \otimes M$ is the tensor product.
Multiplication on $T(M)$ goes as the following: $(v_0, v_1, ...) \times (w_0, w_1, ...) = (..., \sum_{k=0}^{n} v_k \otimes w_{n-k}, ...)$. (In case any of the indices are zero, this becomes the action of R on M). Bilinearity follows from that of tensor multiplication (as well as that of the $R \times M$ operation).
Moreover, $T(M)$ is a *graded algebra*: This is due to the fact that $M^{\otimes k} \otimes M^{\otimes l} \subseteq M^{\otimes k+l}$.

Definition (Universal enveloping algebra)**.** *Let $\mathfrak{g}$ be a Lie algebra over R. Then, the* ***universal enveloping algebra*** *$U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the ideal generated by the relation $i([x, y]) = i(x)i(y) - i(y)i(x)$, where $i : \mathfrak{g} \to T(\mathfrak{g})$ is the obvious inclusion.*

Remark. Alternatively, $U\mathfrak{g}$ is the free algebra on generators $i(x), x \in \mathfrak{g}$, subject to the R-module relations as well as the Lie algebra relation:

1. $i(\alpha x) = \alpha i(x)$
2. $i(x + y) = i(x) + i(y)$
3. $i([x, y]) = i(x)i(y) - i(y)i(x)$

$\alpha \in R, x, y \in \mathfrak{g}$.
Finally, note that $U\mathfrak{g}$ is a *filtered algebra*. The associated graded algebra $\mathfrak{G}(U\mathfrak{g})$ is nothing but the symmetric algebra (which we can think of as the polynomial ring on the basis of $\mathfrak{g}$).

Theorem 3.6. *There is a natural isomorphism between the category of (left) $\mathfrak{g}$-modules and (left) $U\mathfrak{g}$-modules.*

Proof. We describe the correspondence between the two. Let M be a $\mathfrak{g}$-module, and consider a monomial $x_1 \cdot \ldots \cdot x_n \in U\mathfrak{g}$. Then, the formula $(x_1 \cdot \ldots \cdot x_n)m = x_1(x_2(\ldots(x_n m)))$, when extended linearly, will turn M into a $U\mathfrak{g}$-module.
Conversely, let M be a $U\mathfrak{g}$-module. Then, $xm := i(x)m$ turns it into a $\mathfrak{g}$-module (via the relation used to define $U\mathfrak{g}$). $\square$

Theorem 3.7. *Let $U : \mathfrak{g} \mapsto U\mathfrak{g}$ be a functor from Lie algebras to associative algebras, and $\mathrm{Lie} : A \mapsto \mathrm{Lie}(A)$ be a functor from associative algebras to Lie algebras (where we obtain $\mathrm{Lie}(A)$ by defining $[x, y] := xy - yx$ on A).*
Then, U is the left adjoint of Lie.

Proof. We want to show that there is a natural isomorphism $\mathrm{Hom}(\mathfrak{g}, \mathrm{Lie}(A)) \cong \mathrm{Hom}(U\mathfrak{g}, A)$.
First, let $\varphi : \mathfrak{g} \to \mathrm{Lie}(A)$ be a Lie algebra homomorphism. We can extend this linearly to an algebra homomorphism $\overline{\varphi} : U\mathfrak{g} \to A$ in the obvious manner. This gives us a map $\mathrm{Hom}(\mathfrak{g}, \mathrm{Lie}(A)) \to \mathrm{Hom}(U\mathfrak{g}, A), \varphi \mapsto \overline{\varphi}$.
On the other hand, suppose we have an algebra homomorphism $\psi : U\mathfrak{g} \to A$. Then, we can define $\overline{\psi} : \mathfrak{g} \to \mathrm{Lie}(A)$ as $\psi \circ i$. This is obviously an algebra homomorphism; but it is, in fact, also a Lie algebra homomorphism by virtue of the defining relation of $U\mathfrak{g}$.
Naturality and $\overline{\overline{\varphi}} = \varphi, \overline{\overline{\psi}} = \psi$ are now immediate from definition. (Note that the Lie functor will basically do nothing to morphisms, and that U will linearly extend them.) $\square$

Theorem 3.8 (Poincare-Birkhoff-Witt)**.** *Let $\mathfrak{g}$ be a free R-module with basis $\{e_\alpha\}$.*
Then, $U\mathfrak{g}$ is also a free R-module with basis $\{e_I\}$, where I is any increasing sequence of indices from α.

Definition (Free Lie algebra)**.** *Let X be a set and $i : X \to L$ be a function from X into a Lie algebra L. The Lie algebra L is called free on X if, for any Lie algebra A with a function $f : X \to A$, there is a unique Lie algebra homomorphism $g : L \to A$ such that $f = g \circ i$.*

Discussion. A free Lie algebra generated by a set X is the Lie algebra generated by it without any imposed relations other than the defining relations of alternating bilinearity and the Jacobi identity. We shall elaborate a construction of it.
Let $F(X)$ be the free associative algebra on X (canonically isomorphic to the tensor algebra on RX, the free module generated by X with underlying ring R). Let $U(F(X)) = \mathfrak{L}$ be the underlying Lie algebra of the free associative algebra (that is, the one obtained by forgetting all multiplication except the commutator).
Consider the subalgebra $\mathfrak{f}$ of $\mathfrak{L}$ that is generated by X (to be precise, by the first tensor power RX of $T(RX)$—but this is canonically isomorphic to X). We claim that this is the free Lie algebra on X.
Let $f : X \to A$ be a map onto a Lie algebra A, and let UA be its universal enveloping algebra. We have seen that $A \subseteq UA$, and so may thereby recast f as a map from $X \to UA$. If we extend this, we get a homomorphism $\tilde{f} : F(X) \to UA$. This will restrict to a map $g : \mathfrak{f} \to A$. g will be a Lie algebra homomorphism, because $\tilde{f}$ was a homomorphism to begin with. Uniqueness is obvious. This establishes what we wanted.

Lemma 3.9. *Let $\mathfrak{f}$ be the free Lie algebra on the set X. Then, $U\mathfrak{f}$ is the free associative algebra on X.*

Proof. The proof is five steps, culminating in an application of Yoneda's lemma. Let A be any associative algebra on the underlying ring.

- $\mathrm{Hom}_{Alg}(U\mathfrak{f}, A) \cong \mathrm{Hom}_{Lie}(\mathfrak{f}, \mathrm{Lie}(A))$: This follows from left adjointness of U with Lie.
- $\mathrm{Hom}_{Lie}(\mathfrak{f}, \mathrm{Lie}(A)) \cong \mathrm{Hom}_{Set}(X, U(A))$: This follows from the left adjointness of the free Lie algebra functor (on sets) with the forgetful functor.
- $\mathrm{Hom}_{Set}(X, U(A)) \cong \mathrm{Hom}_{Ab}(G(X), A)$: This follows from left adjointness of the free abelian group functor (on sets) with the forgetful functor. (Here, we view A as an abelian group; so, without the multiplicative structure.)
- $\mathrm{Hom}_{Ab}(G(X), A) \cong \mathrm{Hom}_{Alg}(T(G(X)), A)$: This follows from left adjointness of the tensor algebra functor with the forgetful functor (i.e., forgetting the multiplicative structure).

So finally, we have $\mathrm{Hom}_{Alg}(U\mathfrak{f}, A) \cong \mathrm{Hom}_{Alg}(R\{X\}, A)$ for all algebras A. It then follows by Yoneda's lemma that $U\mathfrak{f} \cong R\{X\}$. □

Definition. *Let us denote by F' the ideal $M \oplus (M \otimes M) \oplus ...$ of F, where $M = RX$. We denote by σ the linear map $(x_{i_1} \cdot ... \cdot x_{i_m}) \mapsto [...[x_{i_1} x_{i_2}]....x_{i_m}]$ from $T' \to \mathfrak{f}$ (for $m = 1$, it is identity).*

Theorem 3.10 (Dynkin-Specht-Wever)**.** *Let X be a set and $a \in F$ be a homogeneous element, where F is the free algebra on X. Then, $a \in \mathfrak{f} \iff \sigma(a) = ma$, where m is the degree of a.*

Definition. *We define two functors from $\mathrm{Mod}_{\mathfrak{g}}$ to Mod_R.*

1. $-^{\mathfrak{g}} : M \mapsto M^{\mathfrak{g}} = \{m \in M : xm = 0 \forall x \in \mathfrak{g}\}$. *This is called the invariant submodule.*
2. $-_{\mathfrak{g}} : M \mapsto M_{\mathfrak{g}} = M/\mathfrak{g}M$. *This is called the coinvariant submodule.*

Lemma 3.11. *$-^{\mathfrak{g}}$ is left exact, and $-_{\mathfrak{g}}$ is right exact.*

Proof. Consider the following functor from Mod_R to $\mathrm{Mod}_{\mathfrak{g}}$: Given an R-module M, it turns it into a (trivial) $\mathfrak{g}-$module by having $xm = 0$ for all $x \in \mathfrak{g}, m \in M$.
We will show that $-^{\mathfrak{g}}$ is right adjoint to this trivial functor (henceforth denoted by T), and that $-_{\mathfrak{g}}$ is left adjoint to it. The result will then follow from theorem 1.4.

1. $-^{\mathfrak{g}}$: We want to show that $\mathrm{Hom}(T(M), L) \cong \mathrm{Hom}(M, L^{\mathfrak{g}})$.
   - Let $f : T(M) \to L$ be a $\mathfrak{g}$-module homomorphism (where we denote by L a $\mathfrak{g}$-module). This existence assumption already implies something rather special about L.
     For any $x \in \mathfrak{g}, x \cdot f(m) = f(xm) = f(0)$ (by definition of $T(M)) = 0 = x \cdot l \implies l \in L^{\mathfrak{g}}$. Thus, $L = L^{\mathfrak{g}}$, and we may legitimately define $\overline{f} : M \to L^{\mathfrak{g}}, \overline{f}(m) := f(m)$.

- Let $g : M \to L^{\mathfrak{g}}$ be an R-module homomorphism. There is an obvious way to extend it to a $\mathfrak{g}$-module homomorphism $\overline{g} : T(M) \to L$, which should suffice.

2. $-_{\mathfrak{g}}$: We want to show that $\mathrm{Hom}(L_{\mathfrak{g}}, M) \cong \mathrm{Hom}(L, T(M))$.

   - Let $f : L_{\mathfrak{g}} \to M$ be an R-module homomorphism. Define $\overline{f} : L \to T(M), l \mapsto f([l])$. We need to ensure that this is in accordance with the additional structure required to be a$\mathfrak{g}$-module homomorphism. Firstly, note that $x\overline{f}(l) = 0$ for all $x \in \mathfrak{g}$, by definition of $T(M)$. Now, $\overline{f}(xl) = f([xl]) = f([0])$ (since $xl \in \mathfrak{g}L) = 0$. Therefore, this is a valid map of morphisms.
   - Let $g : L \to T(M)$ be a $\mathfrak{g}$-module homomorphism. The intuitive thing to do here is to define $\overline{g} : L_{\mathfrak{g}} \to M, [l] \mapsto g(l)$. However, one needs to ensure this is well-defined. Suppose $[l] = [l'] \implies l - l' \in \mathfrak{g}L \implies l - l' = x\tilde{l}, x \in \mathfrak{g}, \tilde{l} \in L$. Then, $g(l) - g(l') = g(l - l') = g(x\tilde{l}) = xg(\tilde{l}) = 0$ (since we're now in $T(M)$), and thus, $\overline{g}([l]) = \overline{g}([l'])$, and our map is well-defined.

In both cases, bijectivity and naturality can be quickly checked. □

Homology & Cohomology

Definition (Homology & cohomology groups of a Lie algebra)**.** *Let M be a $\mathfrak{g}$−module.*

- *$L_*(-_{\mathfrak{g}})(M)$ shall be called the homology groups of $\mathfrak{g}$ with coefficients in M.*
- *$R_*(-^{\mathfrak{g}})(M)$ shall be called the cohomology groups of $\mathfrak{g}$ with coefficients in M.*

Remark. Strictly speaking, these are not groups but modules.

Example. Let $\mathfrak{g}$ be the free R-module on the basis $\{e_1, ..., e_n\}$, and given the zero Lie bracket. We will compute its homology and cohomology groups over an arbitrary $\mathfrak{g}$−module M.
Firstly, note that M can equivalently be viewed as a module over the polynomial ring $R[e_1, ..., e_n] = k$.
Next, let us look at R as the trivial $\mathfrak{g}$−module.

- $M_{\mathfrak{g}} = R \otimes_k M$: Using the fact that $R \cong k/ < e_1, ..., e_n > \equiv k/\mathfrak{g}$ and the property $R/I \otimes_R M \cong M/IM$, we get the equality.
- $M^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(R, M)$: If given that R is unital, this is clear from adjointness $(\mathrm{Hom}_{\mathfrak{g}}(R, M) \cong \mathrm{Hom}_R(R, M^{\mathfrak{g}}))$.

From this, it follows that $H_*(\mathfrak{g}, M) \cong \mathrm{Tor}_*^k(R, M); H^*(\mathfrak{g}, M) \cong \mathrm{Ext}_k^*(R, M)$.

Remark. Note that $U\mathfrak{g} = R[e_1, ..., e_n]$. In fact, a generalization of the above example holds.

Theorem 4.12. *Let M be a $\mathfrak{g}$-module. Then:*

$$H_*(\mathfrak{g}, M) \cong \mathrm{Tor}_*^{U\mathfrak{g}}(R, M)$$

$$H^*(\mathfrak{g}, M) \cong \mathrm{Ext}_{U\mathfrak{g}}^*(R, M)$$

Example. Let $\mathfrak{f}$ be any free Lie algebra on a set X. By either reasoning as in the above example or using the fact that $U\mathfrak{f} \cong R\{X\} = k$ (the noncommutative polynomial ring on X) in combination with theorem 3.9, we once again arrive at the results $H_*(\mathfrak{f}, M) \cong \mathrm{Tor}_*^k(R, M); H^*(\mathfrak{f}, M) \cong \mathrm{Ext}_k^*(R, M)$.
Let us now actually compute the Tor and Ext groups.
For this, we first need a projective resolution of R as a k−module. We claim that this is given by the following:

$$0 \to J \to R\{X\} \to R \to 0$$

where $J = XR\{X\}$.
Exactness is clear; so is the fact that $R\{X\} = k$ is projective. From the additional fact that J is free with basis X, we are done, and thus have a projective resolution with us.

Example (*continued*). Continuing the process, we want the homology of the following:

$$0 \to J \otimes_k M \to M \to 0$$

and the homology of the following:

$$0 \to M \to \mathrm{Hom}_{\mathfrak{f}}(J, M) \to 0$$

We thus finally have the following:

- $H^n_{\mathfrak{f}}(\mathfrak{f}, M) = H^{\mathfrak{f}}_n(\mathfrak{f}, M) = 0$ for all $n \geq 2$
- $H^0_{\mathfrak{f}}(\mathfrak{f}, R) = H^{\mathfrak{f}}_0(\mathfrak{f}, R) = R$
- $H^1_{\mathfrak{f}}(\mathfrak{f}, R) = \Pi_{x\in X} R.$
- $H^{\mathfrak{f}}_1(\mathfrak{f}, R) = \bigoplus_{x\in X} R$

where the last three use the facts that, for $M = R$, the differential maps are all zero; and that J is free with basis X.

Definition (Augmentation ideal)**.** *Let $\mathfrak{g}$ be a Lie algebra over R. Consider the k-algebra homomorphism $\epsilon : U\mathfrak{g} \to k, i(\mathfrak{g}) \mapsto 0$. We define the **augmentation ideal** as $\mathfrak{J} := \ker(\epsilon)$.*

Lemma 4.13. $\mathfrak{J}/\mathfrak{J}^2 \cong \mathfrak{g}^{ab} := \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Proof. We construct an isomorphism between the two.

1. It is clear that the inclusion $i : \mathfrak{g} \to U\mathfrak{g}$ maps $[\mathfrak{g}, \mathfrak{g}]$ to $\mathfrak{J}^2$ (by virtue of the defining relation on $U\mathfrak{g}$). Thus, we have an induced map $i : \mathfrak{g}^{ab} \to \mathfrak{J}/\mathfrak{J}^2$.
2. Consider the map $j : U\mathfrak{g} \to \mathfrak{g}^{ab}, i(x) \mapsto \overline{x}$. Clearly, anything in $\mathfrak{J}^2$ will land in $[\mathfrak{g}, \mathfrak{g}]$. Therefore, restricting and quotienting will give us a map $j : \mathfrak{J}/\mathfrak{J}^2 \to \mathfrak{g}^{ab}$.

One can check that $j \circ i, i \circ j$ are identity, which produces the desired conclusion. □

Theorem 4.14. *For any Lie algebra $\mathfrak{g}, H_1(\mathfrak{g}, R) \cong \mathfrak{g}^{ab}$.*

Proof. We start with the following sequence of $\mathfrak{g}$-modules:

$$0 \to \mathfrak{J} \to U\mathfrak{g} \to R \to 0$$

The maps are the obvious ones (inclusion and projection). Exactness is more or less immediate. If we apply Tor and use the fact that $\mathrm{Tor}^{U\mathfrak{g}}_*(U\mathfrak{g}, M) = 0$, we get the following exact sequence:

$$0 \to H_1(\mathfrak{g}, M) \to \mathfrak{J} \otimes_{U\mathfrak{g}} M \to M \to M_{\mathfrak{g}} \to 0$$

(Among other things, we have used theorem 3.9 for the second term in the sequence.)
Set $M = R$. Now, $\mathfrak{J} \otimes_{U\mathfrak{g}} R = \mathfrak{J} \otimes_{U\mathfrak{g}} (U\mathfrak{g}/\mathfrak{J}) \cong \mathfrak{J}/\mathfrak{J}^2$ (by property of tensor product) $\cong \mathfrak{g}^{ab}$ (by lemma 3.10).
Observing that $R \cong R_{\mathfrak{g}}$ and then playing around with the exactness of the sequence a bit, we will ultimately get $H_1(\mathfrak{g}, M) \cong \mathfrak{g}^{ab}$. □

Corollary 4.14.1. *For any trivial $\mathfrak{g}-$module M, $H_1(\mathfrak{g}, M) \cong \mathfrak{g}^{ab} \otimes_R M$.*

Proof. Since $M = M_{\mathfrak{g}}, H_1(\mathfrak{g}, M) \cong \mathfrak{J} \otimes_{U\mathfrak{g}} M \cong (\mathfrak{J} \otimes_{U\mathfrak{g}} R) \otimes_R M \cong \mathfrak{g}^{ab} \otimes_R M$. □

Remark. For $n \geq 2, H_n(\mathfrak{g}, M) \cong \mathrm{Tor}_n^{U\mathfrak{g}}(R, M) \cong \mathrm{Tor}_{n-1}^{U\mathfrak{g}}(\mathfrak{J}, M)$.

Definition (Derivations)**.** *Let M be a $\mathfrak{g}-$module. A derivation is an R-linear map $D : \mathfrak{g} \to M$ such that $D([x, y]) = x(Dy) - y(Dx)$.*

Remark. Derivations are module homomorphisms. Their set $\mathrm{Der}(\mathfrak{g}, M)$ is an R-submodule of $\mathrm{Hom}_R(\mathfrak{g}, M)$.
We can define maps $D_m : x \mapsto xm, m \in M, x \in \mathfrak{g}$. These are also derivations (by module structure on M), called the *inner derivations* and form a further submodule $\mathrm{Der}_{Inn}(\mathfrak{g}, M)$.

Lemma 4.15. $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \cong \mathrm{Der}(\mathfrak{g}, M)$

Proof. Let $\phi : \mathfrak{J} \to M$. Define $D_\phi : \mathfrak{g} \to M, x \mapsto \phi(i(x))$. It is easy to check that D_ϕ is a derivation. We now claim that $\phi \mapsto D_\phi$ is a natural isomorphism. □

Theorem 4.16. $H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Der}_{Inn}(\mathfrak{g}, M)$.

Proof. Once again, we start with the following exact sequence of Lie algebra modules:

$$0 \to \mathfrak{J} \to U\mathfrak{g} \to R \to 0$$

Applying Ext, we get the following exact sequence:

$0 \to \mathrm{Hom}_{U\mathfrak{g}}(R, M) \to \mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g}, M) \to \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{J}, M) \to \mathrm{Ext}^1_{U\mathfrak{g}}(R, M) \to \mathrm{Ext}^1_{U\mathfrak{g}}(U\mathfrak{g}, M) \to ...$

Now, we use the following facts:

- $\mathrm{Ext}_*^{U\mathfrak{g}}(U\mathfrak{g}, M) = 0$
- $\mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{J}, M) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$
- $\mathrm{Ext}^1_{U\mathfrak{g}}(R, M) \cong H^1(\mathfrak{g}, M)$
- $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \cong \mathrm{Der}(\mathfrak{g}, M)$
- $\mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g}, M) \cong M$
- $\mathrm{Hom}_{U\mathfrak{g}}(R, M) \cong \mathrm{Hom}_{\mathfrak{g}}(R, M) \cong M^{\mathfrak{g}}$

to finally end up with this exact sequence:

$$0 \to M^{\mathfrak{g}} \to M \to \mathrm{Der}(\mathfrak{g}, M) \to H^1(\mathfrak{g}, M) \to 0$$

If we show that the kernel of the last map is $\mathrm{Der}_{Inn}(\mathfrak{g}, M)$, we are done. But by exactness, this amounts to showing that the image of $M \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$ is $\mathrm{Der}_{Inn}(\mathfrak{g}, M)$.
Let $m \mapsto \phi, \phi : \mathfrak{J} \to M$. Now, ϕ can be extended to $U\mathfrak{g}$ by defining $phi(1) = m'$. In this case, $D_\phi(x) = \phi(i(x)) = \phi(x \cdot 1) = x \cdot m = D_m(x)$, which establishes that D_ϕ is an inner derivation and thus completes the proof. □

Corollary 4.16.1. *For any trivial* $\mathfrak{g}-$*module* M, $H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M) \cong \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, M) \cong \mathrm{Hom}_R(\mathfrak{g}^{ab}, M)$.

Proof. It is clear that if M is trivial, it can have only trivial inner derivations, which establishes the first equality.
$\mathrm{Der}(\mathfrak{g}, M) \cong \mathrm{Hom}_R(\mathfrak{g}^{ab}, M)$: This is because $D([x, y]) = 0$, so that it vanishes on $[\mathfrak{g}, \mathfrak{g}]$.
The second equality follows by imposing the trivial Lie algebra module structure on Der. □

Remark. For $n \geq 2, H^n(\mathfrak{g}, M) \cong \mathrm{Ext}^{n-1}_{U\mathfrak{g}}(\mathfrak{I}, M)$.

Definition (Lie algebra extension)**.** *An extension* $\mathfrak{e}$ *of a Lie algebra* $\mathfrak{g}$ *by* M *is a short exact sequence of Lie algebras*

$$0 \to M \xrightarrow{i} \mathfrak{e} \xrightarrow{s} \mathfrak{g} \to 0$$

where M *is an abelian Lie algebra.*

Remark. M is a $\mathfrak{g}-$module by defining $gm := i^{-1}([s^{-1}(g), i(m)])$.
Two things to note here are:

- $[s^{-1}(g), i(m)]$ is in the image of i because $i(M) = \mathrm{Ker}(s)$ is an ideal of $\mathfrak{e}$.
- This definition is independent of the choice of inverse. For suppose we have two elements $g_1, g_2 \in s^{-1}(g)$, and suppose further that we have m_1, m_2 such that $i(m_1) = [g_1, i(m)], i(m_2) = [g_2, i(m)]$. Then, $i(m_2 - m_1) = [g_2 - g_1, i(m)]$. But note that $s(g_2 - g_1) = 0 \implies g_2 - g_1 \in \mathrm{Ker}(s) = \mathrm{Im}(M)$. Thus, $i(m_2 - m_1) = [i(\tilde{m}), i(m)] = 0$ (since M is abelian) $\implies m_2 = m_1$.

Definition. *We say two extensions* $0 \to M \to \mathfrak{e}_i \to \mathfrak{g} \to 0$ *are **equivalent** if there is an isomorphism* $\varphi : \mathfrak{e}_1 \to \mathfrak{e}_2$ *such that the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathfrak{e}_1 \arrow[r] \arrow[d, "\varphi"'] & \mathfrak{g} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathfrak{e}_2 \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Let M *be a given* $\mathfrak{g}-$*module. Then, denote by* $\mathrm{Ext}(\mathfrak{g}, M)$ *the set of equivalence classes of extensions which recover the given module structure on* M.

Theorem 4.17 (Classification theorem)**.** *Let* M *be a* $\mathfrak{g}-$*module. Then,* $\mathrm{Ext}(\mathfrak{g}, M)$ *is in 1-1 correspondence with* $H^2(\mathfrak{g}, M)$.

The Chevalley-Eilenberg complex

Definition (Modules)**.** *Let $\mathfrak{g}$ be a Lie algebra over R that is free as an R-module. Let $\Lambda^p\mathfrak{g}$ its p^{th} exterior product. Then, we define $V_p(\mathfrak{g}) := U\mathfrak{g} \otimes_R \Lambda^p\mathfrak{g}$.*

Remark. $V_p(\mathfrak{g})$ is free as a $U\mathfrak{g}$-module, with basis $1 \otimes B$ (where B is the basis of $\Lambda^p\mathfrak{g}$).

Definition (Differentials)**.** *Let $d : V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ be defined by $d(u \otimes x_1 \wedge x_2 ... \wedge x_p) = \theta_1 + \theta_2$, where (for $u \in U\mathfrak{g}, x_i \in \mathfrak{g}$)*

$$\theta_1 = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \wedge ... \hat{x_i} \wedge ... \wedge x_p$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge ... \wedge \hat{x_i} \wedge ... \wedge \hat{x_j} ... \wedge x_p$$

Remark. We separately define $d : V_1(\mathfrak{g}) \to V_0(\mathfrak{g})$ as $u \otimes x \mapsto ux$ (using $\Lambda^0\mathfrak{g} = R, \Lambda^1\mathfrak{g} = \mathfrak{g}$).

Lemma 5.18. $d^2 = 0$.

Proof. If we write $d(\theta_i) = \theta_{i1} + \theta_{i2}$, it will turn out that $\theta_{22} = 0, -\theta_{11}$ is the $i = 1$ part of θ_{21} and $-\theta_{12}$ is the $i > 1$ part of θ_{21}, so that the sum vanishes. ☐

Definition (The Chevalley-Eilenberg complex)**.** *$V_*(\mathfrak{g})$, with the given differential, is a chain complex of $U\mathfrak{g}-$modules. This is called the* ***Chevalley-Eilenberg complex****.*

Theorem 5.19. *$V_*(\mathfrak{g}) \xrightarrow{\epsilon} R$ is a projective resolution of the $\mathfrak{g}-$module R.*

Corollary 5.19.1 (Chevalley-Eilenberg)**.** *If M is a right $\mathfrak{g}-$module, then $H_*(\mathfrak{g}, M)$ are the homology of the chain complex $M \otimes_R \Lambda^*\mathfrak{g}$.*
If M is a left $\mathfrak{g}-$module, then $H^*(\mathfrak{g}, M)$ are the cohomology of the cochain complex $\mathrm{Hom}_R(\Lambda^*\mathfrak{g}, M)$.

Proof. We have seen that $V_*(\mathfrak{g}) \xrightarrow{\epsilon} R$ is a projective resolution of the $\mathfrak{g}-$module R. Therefore, $H_n(M \otimes_{U\mathfrak{g}} V_*(\mathfrak{g})) \cong \mathrm{Tor}_n^{U\mathfrak{g}}(R, M) \cong H_n(\mathfrak{g}, M)$ (theorem 3.10). But also, $M \otimes_{U\mathfrak{g}} V_n(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_R \Lambda^*\mathfrak{g} = M \otimes_R \Lambda^*\mathfrak{g}$.
Similarly, $H^n(\mathrm{Hom}_\mathfrak{g}(V_*(\mathfrak{g}), M)) \cong \mathrm{Ext}_n^{U\mathfrak{g}}(R, M) \cong H^n(\mathfrak{g}, M)$. And also, $\mathrm{Hom}_\mathfrak{g}(V_*(\mathfrak{g}), M) = \mathrm{Hom}_\mathfrak{g}(U\mathfrak{g} \otimes_R \Lambda^*\mathfrak{g}, M) \cong \mathrm{Hom}_R(\Lambda^*\mathfrak{g}, \mathrm{Hom}_\mathfrak{g}(U\mathfrak{g}, M)) \cong \mathrm{Hom}_R(\Lambda^*\mathfrak{g}, M)$ (where we have used tensor-hom adjunction at the end). ☐

Remark. In the cochain complex, an element f of $\mathrm{Hom}_R(\Lambda^*\mathfrak{g}, M)$ is an alternating R-multilinear map of n variables in $\mathfrak{g}$ taking values in M.
Its coboundary δf is the $(n+1)$ cochain $\delta f(x_1, ..., x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} x_i f(x_1, ..., \hat{x_i}, ... x_{n+1}) + \sum_{i<j} (-1)^{i+j} f([x_i, x_j], x_1, ..., \hat{x_i}, ..., \hat{x_j}, ..., x_{n+1})$.

Postscript

$\mathbb{Z}$-Graded Lie superalgebras

In order to remain consistent with the rational homotopy theory literature, we shall henceforth just say 'graded Lie algebra' for what follows.

Definition. *A **graded Lie algebra** L is a graded vector space $\{L_i\}_{i\in\mathbb{Z}}$ equipped with a linear map of degree zero $L \times L \to L, (x,y) \mapsto [x,y]$ such that*

1. $[x,y] = -(-1)^{ij}[y,x]$ *(Antisymmetry)*
2. $(-1)^{ik}[x,[y,z]] + (-1)^{jk}[z,[x,y]] + (-1)^{ij}[y,[z,x]] = 0$ *(Jacobi identity)*

where i,j,k are the degrees of x,y,z respectively.

*A **morphism** of graded Lie algebras is a map $\phi : L \to M$ such that $\phi(L_i) \subseteq M_i$ and $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$.*

Definition. *Let L be a graded Lie algebra. A (left) L-module is a graded vector space V equipped with a map $\psi : L \times V \to V, (l,v) \to l \cdot v$ such that $[x,y]\cdot v = y\cdot(x\cdot v) - (-1)^{ij} x \cdot (y \cdot v)$.*

Remark. Like in the case of ordinary Lie algebras, a module can also be described with a graded Lie algebra morphism between L and $\mathrm{Hom}(V,V)$, where L is a graded Lie algebra and V is a graded vector space (so that its Homset, when equipped with the bracket $[x,y] \mapsto xy - (-1)^{ij}yx$, is a graded Lie algebra—recall that its gradation will come from the degree of the linear map).

Hopf algebras

Definition (Coalgebra)**.** *A **coalgebra** over a field K is a vector space C over K together with maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to K$ such that the following diagrams commute:*

1. $$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{Id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{Id}"] & C \otimes C \otimes C \end{tikzcd}$$

2. $$\begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[d, "\Delta"] \arrow[dr, "\mathrm{Id}"] & & C \otimes C \arrow[d, "\mathrm{Id}\otimes\epsilon"] \\ C \otimes C \arrow[r, "\epsilon\otimes\mathrm{Id}"] & K \otimes C \cong C & \cong C \otimes K \end{tikzcd}$$

Remark. The first diagram is the dual of the one expressing associativity of algebra multiplication (called the coassociativity of the comultiplication); the second diagram is the dual of the one expressing the existence of a multiplicative identity. The map Δ is called the comultiplication (or coproduct) of C and ϵ is the counit of C.

Definition (Bialgebra)**.** *A **bialgebra** is a vector space C over a field K equipped with both a unital associative algebra structure, as well as with a counital coassociative algebra sructure, satisfying the following compatibility conditions (where ∇ is the multiplication and η is the unit):*

1.
$$\begin{tikzcd} B \otimes B \arrow[r, "\nabla"] \arrow[d, "\Delta\otimes\Delta"] & B \arrow[r, "\Delta"] & B \otimes B \\ B \otimes B \otimes B \otimes B \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] & & B \otimes B \otimes B \otimes B \arrow[u, "\nabla\otimes\nabla"'] \end{tikzcd}$$

where $\tau : x \otimes y \mapsto y \otimes x$

2.
$$\begin{tikzcd} B \otimes B \arrow[rr, "\nabla"] \arrow[dr, "\epsilon\otimes\epsilon"] & & B \arrow[dl, "\epsilon"] \\ & K \otimes K \cong K & \end{tikzcd}$$

3.
$$\begin{tikzcd} & K \otimes K \cong K \arrow[dl, "\eta\otimes\eta"'] \arrow[dr, "\eta"] & \\ B \otimes B & & B \arrow[ll, "\Delta"] \end{tikzcd}$$

4.
$$\begin{tikzcd} K \arrow[dd, "\mathrm{Id}"] \arrow[dr, "\eta"] & \\ & B \arrow[dl, "\epsilon"] \\ K & \end{tikzcd}$$

Definition (Hopf algebra)**.** *A **Hopf algebra** H is a bialgebra equipped with a K-linear map $S : H \to H$ (called the antipode) such that the following diagram commutes:*

$$\begin{tikzcd} H \otimes H \arrow[rr, "S\otimes\mathrm{Id}"] & & H \otimes H \arrow[d, "\nabla"] \\ H \arrow[u, "\Delta"] \arrow[d, "\Delta"] \arrow[r, "\epsilon"] & K \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "\mathrm{Id}\otimes S"] & & H \otimes H \arrow[u, "\nabla"'] \end{tikzcd}$$

Remark. A **group-like element** is an $x \in H$ such that $\Delta(x) = x \otimes x$. A **primitive element** is an $x \in H$ such that $\Delta(x) = 1 \otimes x + x \otimes 1$.

Example. The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is an example of a Hopf algebra, with $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$ (and being extended linearly elsewhere), $\epsilon : x \mapsto 0$, and $S : x \mapsto -x$.

Discussion. Let $\mathbb{H}$ denote the category of Hopf algebras, and $\mathbb{L}$ the category of Lie algebras. We have a functor $U : \mathbb{L} \to \mathbb{H}$, sending a Lie algebra to its universal enveloping algebra, and a functor $P : \mathbb{H} \to \mathbb{L}$, sending a Hopf algebra to the Lie algebra of its primitive elements. The Milnor-Moore theorem states that $U \circ P = \mathrm{Id}$; and a theorem attributed to Friedrichs states that $P \circ U = \mathrm{Id}$.