Differential Geometry

Aditya Dwarkesh

Smooth Manifolds

Definition (Manifold). *A* ***topological manifold*** *M is a second countable Hausdorff space which is* ***locally Euclidean****, that is, given a point $p \in M$, there exists a neighbourhood $p \in U$ such that U is homeomorphic to some open subset of $\mathbb{R}^n$.*

Definition (Atlas). *An* ***atlas*** *is a collection of ordered pairs (U, ϕ) called coordinate charts such that $U \subseteq M$ and $\phi : U \to \tilde{U} \subseteq \mathbb{R}^n$ is a homeomorphism.*

Definition. *Two coordinate charts $(U, \phi), (V, \psi)$ are said to be* ***smoothly compatible*** *if either $U \cap V = \emptyset$ or $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism.*

Definition. *A smooth atlas for a manifold is an atlas such that all the charts are smoothly compatible with one another.*

Lemma 1.1. *Given a smooth atlas A, there is a unique maximal smooth atlas $\tilde{A}$ containing A. We call this the* ***smooth structure*** *on M.*

Remark. For $n \neq 4, \mathbb{R}^n$ has a unique smooth structure (up to diffeomorphism). However, $\mathbb{R}^4$ has uncountably many distinct smooth structures.

Definition (Smoothness). *A function $f : M \to N$ is called* ***smooth*** *at p if there exist coordinate charts $(U, \phi) \subseteq M, (V, \psi) \subseteq N$ such that $f(U) \subseteq V$, and $\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is smooth at $\phi(p)$.*

Tangent Spaces

Definition (Derivation). *Given a smooth manifold M of dimension n, a* ***derivation*** *at a point $p \in M$ is a linear map $X : C^\infty(M) \to \mathbb{R}$ such that $X(fg) = X(f)g(p) + f(p)X(g)$.*

Definition (Tangent Space). *The tangent space of a manifold at a point p, T_pM, is the vector space (over $\mathbb{R}$) of all derivations at p.*

Definition (Pushforward)**.** *Given a smooth map $F : M \to N$, the **differential** of F at p, denoted by dF_p or $F_{*,p} : T_pM \to T_{F(p)}N$, is defined by $dF_p(X)(f) = X(f \circ F)$.*

Theorem 2.1. *$\{\frac{\partial}{\partial e^i}|_a\}$ form a basis for $T_a\mathbb{R}^n$.*

Lemma 2.2. *Let $\frac{\partial}{\partial x^i|_p} := d(\phi^{-1})_{\phi(p)}(\frac{\partial}{\partial e^i}|_{\phi(p)})$, where $\phi : U \to \tilde{U} \subseteq \mathbb{R}^n$ is a homeomorphism (making $d\phi_p : T_pU \cong T_pM \to T_{\phi(p)}U \cong T_{\phi(p)}\mathbb{R}^n$ an isomorphism). Then, $\{\frac{\partial}{\partial x^i}|_p\}$ forms a basis for T_pM.*

Lemma 2.3. *For a smooth map $F : M \to N$, the matrix representation of dF_p is given by $a_{ij} = \frac{\partial F^i}{\partial x^j}|_p := \frac{\partial(\psi \circ F \circ \phi^{-1})^i}{\partial e^j}|_{\phi(p)}$.*

Definition. *Given a curve $\gamma : I \to M$ such that $\gamma(0) = p$, define $\gamma_1 \sim \gamma_2 \iff (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for all $f \in C^\infty(M)$.*

Lemma 2.4. *Given $\gamma : I \to M$, define $\gamma'(0) = d\gamma_0(\frac{d}{dt}|_0) \in T_{\gamma(0)}M$. Then, given $X \in T_pM$, there exists a smooth curve such that $\gamma'(0) = X$.*

Theorem 2.5. *Let V_pM be the vector space constituted by the equivalence classes of curves $[\gamma]$. Then, $V_pM \cong T_pM$ through the mapping $[\gamma] \mapsto \gamma'(0)$.*

Remark. The above theorem offers a more geometric perspective on the tangent space. It is constituted by the set of 'curves' (initialized at the same point) on M pointing in all possible directions (wherein two curves are said to point in the same 'direction' if any function has the same derivative along them).

Definition (Tangent bundle)**.** *$TM = \sqcup_p T_pM$ is known as the **tangent bundle** of a manifold M.*

Theorem 2.6. *TM is a smooth manifold of dimension $2n$.*

Remark. n from the manifold M, and n more from T_pM.

Vector Fields

Definition (Vector field)**.** *A **vector field** on a smooth manifold is a map $X : M \to TM$ such that $\pi \circ X = id_M$. If X is a smooth map, we call it a smooth vector field.*

Remark. X assigns a vector to each point on the manifold. The set of all smooth vector fields on a manifold, denoted by $\chi(M)$, forms a vector space over $\mathbb{R}$. The following theorem characterizes smooth vector fields.

Theorem 3.1. *The following statements are equivalent:*

1. *X is a smooth vector field*
2. *$Xf \in C^\infty(M) \forall f \in C^\infty(M)$, where $(Xf)(p) := (X_p)f$*
3. *$\{f_i\}$ are smooth, where $X = \sum f_i \frac{\partial}{\partial x^i}$.*

Remark. The expression in (3) is obtained by setting $f_i = X^i$, where $X_p = \sum X^i(p) \frac{\partial}{\partial x^i}|_p$ for a vector field X. The expression will be local, since the basis refers to a coordinate neighbourhood of p.
The next theorem offers an identification of elements of the vector space $\chi(M)$ with elements of the ring $C^\infty(M)$. Note that, as such, the former forms a module over the latter.

Theorem 3.2. *Let X be any smooth vector field and $y : C^\infty(M) \to C^\infty(M)$ be any derivation. Then, the following hold:*

1. *$X \in C^\infty(M)$*
2. *There exists a $Y \in \chi(M)$ such that $Yf = yf$ for all $f \in C^\infty(M)$*

Definition (F-related)**.** *Let $F : M \to N$ be a smooth map. Then, $X \in \chi(M)$ is said to be **F-related** to $Y \in \chi(N)$ if $dF_p(X_p) = Y_{F(p)}$ for all $p \in M$.*

Remark. The next lemma offers an equivalent characterization of F-relatedness.

Lemma 3.3. *Let $F : M \to N$ be smooth. Then, $X \in \chi(M), Y \in \chi(N)$ are F-related $\iff \forall f \in C^\infty(N), X(f \circ F) = (Yf) \circ F$.*

Remark. The next theorem essentially tells us when the pushforward of a vector field F_*X (defined as $F_*(X)(p) = dF_p(X_p)$) is smooth. The condition is slightly stronger than just having F be smooth.

Theorem 3.4. *If $F : M \to N$ is a diffeomorphism, for every $X \in \chi(M)$, there is a unique $Y \in \chi(N)$ that is F-related to X.*

Submersions, Immersions, Submanifolds

Definition (Rank)**.** *The rank of* $F : M \to N$ *at* $p \in M$ *is the rank of the linear map* $dF_p : T_pM \to T_{F(p)}N$*.*

Definition (Submersion/Immersion)**.** *A smooth map* $F : M \to N$ *is said to be a submersion (immersion resp.) if it has constant rank* n *(*m *resp.).*

Definition (Smooth embedding)**.** *A smooth embedding is an immersion homeomorphic on its image.*

Theorem 4.1 (Rank theorem)**.** *Let* $F : M \to N$ *be of constant rank* k*. Given* $p \in M$*, there exists coordinate neighbourhoods around* $p, F(p)$ *with* $F(U) \subseteq V$ *such that* $\hat{F}(p)(u_1, ...u_m) = (u_1, ...u_k, 0, ...0)$*.*

Definition (Submanifold)**.** *Let* M *be a smooth* n*-manifold and* $S \subseteq M$ *be a subset with the subspace topology. Furthermore, suppose* S *is covered by charts* (U, ϕ) *such that* $\phi(S \cap U)$ *is a* k*-slice in* $\mathbb{R}^n$*. Then,* S *is a topological manifold with dimension* k*, and we call it an* ***embedded submanifold****.*

Lemma 4.2. $i : S \to M$ *is a smooth embedding.*

Lemma 4.3. *If* $F : M \to N$ *is a smooth embedding,* $F(M)$ *is an embedded submanifold of* N*.*

Definition. *Let* $\phi : M \to N$ *be a map.*

1. *For any* $c \in N$*, the set* $\phi^{-1}(c) \subseteq M$ *is called a* ***level set*** *of* ϕ*.*
2. *If* $p \in M$ *is such that* $d\phi_p$ *is surjective, we call* p *a* ***regular point*** *of* ϕ*.*
3. *If* $c \in N$ *is such that every* $p \in \phi^{-1}(c)$ *is a regular point, we call* c *a* ***regular value*** *of* ϕ*.*

If something is not a regular point (value resp.), we call it a critical point (value resp.).

Theorem 4.4 (Constant rank level set theorem)**.** *Let* $\Phi : M \to N$ *be a smooth map of constant rank* k*. Then, each level set of* Φ *is an embedded submanifold of* M *with codimension* k*..*

Lemma 4.5. *Let $\phi : M \to N$ be a smooth map of constant rank and S be any level set of ϕ. Then, $T_pS = \ker(d\phi_p)$ for all $p \in S$.*

Differential Forms

Definition (Cotangent space)**.** *A* ***cotangent vector*** *is an element of the* ***cotangent space*** *$(T_pM)^* = T_p^*M$. The* ***cotangent bundle*** *is the collection $T^*M = \sqcup_p T_p^*M$.*

Definition (Differential 1-form)**.** *A* ***differential one-form*** *is a map ω from $M \to T^*M$ such that $\pi \circ \omega = Id_M; p \mapsto \omega_p \in T_p^*M$.*

Definition (Differential of a map)**.** *For $f \in C^\infty(M)$, the* ***differential*** *of f at $p \in M$ is the cotangent vector $df_p : T_pM \to \mathbb{R}$ defined by $df_p(X_p) := X_pf$.*

Remark. We have already used the phrase 'differential of a map' in section 2 to refer to something slightly different. If we use the fact that $T_a\mathbb{R} \cong \mathbb{R}$, it can be seen that they actually amount to the same thing.
The cotangent bundle will be a manifold of dimension $2n$ with constructions largely identical to that of the tangent bundle.

Lemma 5.1. *The dual basis of $\{\frac{\partial}{\partial x_i}|_p\}$ is given by $\{dx_{ip}\}$.*

Remark. Much like the case of vector fields, we can, in some coordinate neighbourhood U, write a one-form as $\omega = \sum_i a_i dx_i$.
The next theorem characterizes smooth one-forms. Observe how the third statement allows us to recast one-forms as maps from $\chi(M)$ to $C^\infty(M)$.

Theorem 5.2. *The following statements are equivalent:*

1. *ω is a smooth one-form*
2. *$\{a_i\}$ are all smooth*
3. *$\omega X \in C^\infty(M)$ for all $X \in \chi(M)$, where $\omega X(p) := \omega_p(X_p)$.*

Definition (Tensors)**.** *A* ***k-tensor*** *is a multilinear map $T : \underbrace{V \times .. \times V}_{k \text{ times}} \to \mathbb{R}$.*
An ***alternating*** *k-tensor is a k-tensor such that $T(v_{\sigma(1)}, ...v_{\sigma(k)}) = sgn(\sigma)T(v_1, ...v_k)$, where $\sigma \in S_k$.*
The set of all alternating k-tensors on a vector space V forms a vector space $\Lambda^k(V)$.

Lemma 5.3. *If V has dimension n, $\Lambda^k(V)$ has dimension nC_k.*

Definition (Exterior bundle)**.** *We call $\Lambda^k(M) = \sqcup_{p\in M}\Lambda^k(T_pM)$ the* ***exterior bundle*** *on M.*

Remark. The exterior bundle of a manifold is itself a smooth manifold of dimension $n + nC_k$ (n from the manifold M, and nC_k more from $\Lambda^k(T_pM)$).

Definition (Differential k-form)**.** *A* ***differential k-form*** *is a map $s : M \to \Lambda^k(M)$ such that $\pi \circ s = Id_M$.*

Remark. Observe how $\Lambda^1(M) = T^*M$, making this consistent with our earlier understanding of one-forms.
We denote the space of smooth $k-$forms on M by $\Omega^k(M)$, and $\Omega^*(M) = \sqcup_k\Omega^k(M)$.

Definition (Pullback)**.** *Let $F : M \to N$ be a smooth map. Then, $F^{*,p} : \Lambda^k(T_{F(p)}N) \to \Lambda^k(T_pM)$, is defined by $(F^{*,p}s)(v_1, ..., v_k) = s(F_{*,p}v_1, ...F_{*,p}v_k)$ (where $s \in \Lambda^k(T_{F(p)}N), v_i \in T_pM$).*

The ***pullback*** *of $F, F^* : \Omega^*(N) \to \Omega^*(M)$, is defined by $(F^*\omega)_p = F^{*,p}\omega_{F(p)}$.*

Definition (Wedge product)**.** *Let $\omega \in \Omega^k(M), \eta \in \Omega^l(M)$. Then, $(\omega \wedge \eta)_p = \omega_p \wedge \eta_p$, where the right-hand side is the antisymmetrization of $\omega_p \cdot \eta_p$.*

Remark. One could imagine the pullback as in contrast with the pushforward, which is as $F_* : \chi(M) \to \chi(N), (F_*X)_p = F_{*,p}X_p$.
The wedge product and pullback of smooth forms is smooth.
$(\Omega^*(M), \wedge)$ forms a graded algebra, and on it, F^* is a graded algebra homomorphism.
By convention, we have $F^*h = h \circ F$ for a C^∞ function h.

Lemma 5.4. *If $\{v_1, ..., v_n\}$ is a basis for V and $\{e_1, ..., e_n\}$ is the dual basis, then $\{e_{i_1} \wedge ... \wedge e_{i_k}\}_{1\leq i_1 ...\leq i_k\leq n}$ is a basis for $\Lambda^k(V)$, and $(e_{i_1} \wedge ... \wedge e_{i_k})(v_1, ..., v_k) = \det[e_{i_l}(v_j)]_{l,j}$.*

Remark. The formula will follow from the definition of the wedge product and the Leibniz formula for determinants. With it, we can write $df_1 \wedge ... \wedge df_k = \sum_{1\leq i_1\leq ...i_k\leq n} a_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}$, where $a_{i_1,...,i_k} \in C^\infty(M)$.
From the same argument, one can write, for $f_i \in C^\infty(M), (df_1 \wedge ... \wedge df_k)(\frac{\partial}{\partial x_{\mu_1}}, ..., \frac{\partial}{\partial x_{\mu_k}}) = \det[\frac{\partial f_i}{\partial x_{\mu_j}}]^k_{i,j=1}$.
In the particular case of top forms, we have the following change-of-coordinates formula:
$dx_1 \wedge ... \wedge dx_n = hdy_1 \wedge ... \wedge dy_n, h = \det[\frac{\partial x_i}{\partial y_j}]^k_{i,j=1}$.

Definition (Anti-derivation)**.** *An* ***anti-derivation*** *(of degree 1) $D : \Omega^*(M) \to \Omega^*(M)$ is an $\mathbb{R}$-linear map such that:*

1. $D(\omega \wedge \eta) = (D\omega) \wedge \eta + (-1)^k\omega \wedge (D\eta), \omega \in \Omega^k(M), \eta \in \Omega^l(M)$
2. $D(\Omega^k(M)) \subseteq \Omega^{k+1}(M)$.

Definition (Exterior derivative)**.** *An **exterior derivative** on M is an antiderivation D on $\Omega^*(M)$ such that:*

1. $D \circ D = 0$
2. $(Df)(X) = Xf$

Theorem 5.5. *Given a smooth manifold M, a unique exterior derivative d exists.*

Remark. We can write out the action of d explicitly.
For $\omega \in \Omega^k(M), dw = d(\sum_{1 \leq i_1 \leq \ldots i_k \leq n} a_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k})$
$= \sum_{1 \leq i_1 \leq \ldots i_k \leq n, 1 \leq j \leq n} \frac{\partial a_{i_1,\ldots,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

Lemma 5.6. *F^* commutes with d.*

Orientation

Definition (Orientation on a vector space)**.** *Let V be an n-dimensional vector space. $\beta \in \Lambda^n(V)$ induces the orientation $[v_1, \ldots, v_n]$ if $\beta(v_1, .., v_n) > 0$, where $(v_1, \ldots, v_n) \sim (u_1, \ldots, u_n) \iff$ the two basis sets are related by a matrix of positive determinant.*

Remark. Alternatively, we can define an equivalence class on $\Lambda^n(V)$ (which, recall, is one-dimensional) by deeming $\beta \sim \eta \iff \beta = c\eta, c > 0$. This allows us to define orientation as an equivalence class of covectors.

Definition (Orientation on a manifold)**.** *A pointwise/rough orientation on M is a collection of orientations $[\mu_p]$ on T_pM.*

Definition (Frame)**.** *A **frame** for an n-dimensional manifold is a collection of vector fields $X_1, \ldots, X_n$ such that for all $p \in M, \{X_{1,p}, \ldots, X_{n,p}\}$ forms a basis for T_pM.*

Definition (Smooth orientation)**.** *A pointwise orientation μ is said to be smooth at p if there exists a frame $X_1, \ldots X_n$ smooth at p such that $[X_{1,p}, \ldots, X_{n,p}] \sim \mu_p$.*

Definition (Orientable)**.** *A manifold M is **orientable** if it admits a smooth orientation.*

Lemma 6.1. *A connected orientable manifold has exactly two orientations.*

Theorem 6.2. *Let M be a smooth n-dimensional manifold. Then, the following statements are equivalent:*

1. *M is orientable*
2. *M admits a nowhere vanishing smooth n-form*
3. *The transition maps on M all have positive Jacobian*

Remark. An orientation can be defined on an orientable manifold by making a choice of nowhere-vanishing top form; top forms can be partitioned into two elements through the equivalence relation $\omega \sim \omega' \iff \omega = f\omega'$ for some $f \in C^\infty(M)$ such that $f > 0$.

Definition (Orientation-preserving maps)**.** *Let $(M, [\omega_M]), (N, [\omega_N])$ be two oriented smooth manifolds, and $F : M \to N$ be a smooth map. We say F is **orientation-preserving** if $[F^*\omega_N] = [\omega_M]$.*

Remark. It can be shown that the third statement in the above theorem amounts to saying precisely that all the transition maps of the manifold are orientation-preserving.

Definition (Contraction)**.** *The map $i_v : \Lambda^k(V) \to \Lambda^{k-1}(V), \omega \mapsto i_v\omega$, called interior multiplication or contraction with v, is defined as $i_v\omega(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1})$.*

Integration

Definition (Domain of integration)**.** *A **domain of integration** $D \subseteq \mathbb{R}^n$ is a bounded set such that ∂D has (n-dimensional) Lebesgue measure zero.*

Definition. *Let ω be a top form on $\mathbb{R}^n$, and D be a domain of integration. Then, $\int_D \omega := \int_D f dV$, where $\omega = f dx_1 \wedge ... \wedge dx_n$ for some smooth function $f : \mathbb{R}^n \to \mathbb{R}$.*

Lemma 7.1. *Let D, E be open domains of integration in $\mathbb{R}^n$ and $G : \overline{D} \to \overline{E}$ be a diffeomorphism. Then, given a top form ω on E, $\int_D G^*\omega = \pm \int_E \omega$ (depending on whether G is orientation-preserving or reversing.)*

Discussion. This fact will prove useful in the following discussion: For any compact subset of an open set $K \subseteq U$, there exists a domain of integration such that $K \subseteq D \subseteq \overline{D} \subseteq U$. Now, let ω be a compactly supported top-form of $\mathbb{R}^n$ whose support is contained in some open set U. Then, $\int_U \omega = \int_D \omega$ (where D is the domain of integration via the above result).

If we restrict ourselves to compactly supported n-forms, lemma 7.1 holds for any open subsets D, E.

Definition. *Let M be a smooth oriented manifold and ω a compactly supported top form within a (positively oriented) chart (U, φ). Then, $\int_M \omega := \int_{\varphi(U)} (\varphi^{-1})^* \omega$.*

Lemma 7.2. *$\int_M \omega$, as defined above, is independent of choice of chart.*

Definition (Integration on manifolds)**.** *Let M be a smooth oriented manifold and ω a compactly supported top form. Let (U_i, φ_i) cover $\operatorname{supp}(\omega)$, and ψ_i be a subordinate partition of unity. Then, we define*

$$\int_M \omega = \sum_i \int_M \psi_i \circ \omega$$

Remark. It can be shown that the above definition is independent of the choice of partition of unity.

Postscript

Lie groups & Lie algebras

Definition (Lie bracket of vector fields)**.** *Let $X, Y \in \chi(M)$. Then, $[X, Y] : C^\infty(M) \to C^\infty(M)$ is defined as $X(Yf) - Y(Xf)$.*

Definition. *A Lie group G is a smooth manifold with group structure such that the map $G \times G \to G, (g_1, g_2) \mapsto g_1 g_2^{-1}$ is smooth.*

Lemma 8.1. *The space $\chi(M)$ equipped with the bracket $[,]$ forms a Lie algebra.*

Lemma 8.2. *Let $F : M \to N$ be a diffeomorphism. If $\tilde{X}, \tilde{Y}$ are F-related to X, Y, then $[\tilde{X}, \tilde{Y}]$ is F-related to $[X, Y]$.*

Definition. *Let G be a Lie group. Then, denote by L_g the diffeomorphism $g' \mapsto gg'$.*

Definition. *$X \in \chi(G)$ is said to be **left-invariant** if it is L_g-related to itself for all $g \in G$; that is, $(dL_g)_{g'}(X_g) = X_{gg'}$.*

Theorem 8.3. *Let $\mathrm{Lie}(G)$ be the space of all left-invariant vector fields on G. Then, $\mathrm{Lie}(G)$ is a Lie algebra, and $\mathrm{Lie}(G) \cong T_e(G)$.*

Boundary

We denote by $\mathbb{H}^n$ the closed upper-half plane, that is, $\{(x_1, ..., x_n) \in \mathbb{R}^n : x^n \geq 0\}$.

Discussion. An n**-dimensional topological manifold with boundary** is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$.
An open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ will be called a **chart** for M.
We will call (U, φ) an **interior chart** if $\varphi(U)$ is an open subset of $\mathbb{R}^n$, and a **boundary chart** if $\varphi(U)$ is an open subset of $\mathbb{H}^n$ such that $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$.
A point $p \in M$ is called an **interior point** if it is in the domain of some interior chart; and a **boundary point** if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$.
The boundary of M (the set of all its boundary points) is denoted by ∂M. It is an $(n-1)$ dimension submanifold without boundary of M.

It is clear that any $p \in M$ is either an interior point or a boundary point. However, it can also be proven that no point can be both in the interior and in the boundary.

We end by stating a landmark result in differential geometry.

Theorem 8.4 (Stokes' theorem)**.** *Let M be a smooth, oriented n-dimension manifold with boundary, and ω be a compactly supported smooth (n-1) form. Then:*

$$\int_M dw = \int_{\partial M} \omega$$