Algebraic Topology

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The Fundamental Group

Results

The Fundamental Group of the Circle

Theorem 1.1. $\pi_1(S^1) \cong \mathbb{Z}$

Remark. We have not mentioned a basepoint because, since the circle is path connected, the same fundamental group will be associated regardless of the choice of point.

Proof. Define $\gamma_n : I \to S^1, t \mapsto e^{2\pi i n t}$. Each γ_n is a loop which goes around the circle n times. Now, define $\Phi : \mathbb{Z} \to \pi_1(S^1), n \mapsto [\gamma_n]$. We claim that Φ is a group isomorphism.

- Homomorphism: We wish to show that $\Phi(m+n) = \Phi(m) * \Phi(n) \iff [\gamma_{m+n}] = [\gamma_m] * [\gamma_n] \iff [\gamma_{m+n}] = [\gamma_m * \gamma_n]$.
  $\gamma_{m+n}(s) = e^{2\pi i(m+n)s}$
  $\gamma_m * \gamma_n(s) = \begin{cases} e^{2\pi i m 2s} & s \in [0, \frac{1}{2}] \\ e^{2\pi i n(2s-1)} & s \in [\frac{1}{2}, 1] \end{cases}$
  Clearly, the two are not equal as maps. We shall have to construct a nontrivial homotopy between them.
  For this, it suffices to show that $\gamma_n \sim \underbrace{\gamma_1 * (\gamma_1 * ...(\gamma_1 * \gamma_1))}_{\text{n times}}$.
  We shall prove this by induction. $n = 1, 2$ are obvious from definition. Suppose this is true for $n = k$. We now need to show that $\gamma_{k+1} \sim \gamma_1 * \gamma_k$.
  The $k+1$ divisions on the upper horizontal represent the number of loops run by γ_{k+1}.

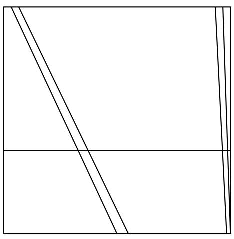

Figure 1: The Square of Homotopy

  On the lower horizontal, which shows us $\gamma_1 * \gamma_k$, we run one loop in half the time, and k loops in the remaining half. The straight lines connecting the divisions show us the most natural homotopy between the two.
  Consider the horizontal line drawn at a height t. By Thales' theorem, this will intersect the first line at a distance $f(t) = \frac{1}{k+1} + \frac{1}{2}(1-t)(1 - \frac{2}{k+1})$. So at t, we want to complete our first loop by $s = f(t)$. So for $t < f(t)$, we write $h(s,t) = e^{2\pi i \frac{s}{f(t)}}$.

The remaining k loops will each take $\frac{1-f(t)}{k}$ time. So, for $t \geq f(t)$, we write $h(s,t) = e^{2\pi i k(s-f(t))}$.

Thus, the required homotopy is $h(s,t) = \begin{cases} e^{2\pi i \frac{s}{f(t)}} & t \in [0, f(t)] \\ e^{2\pi i k(s-f(t))} & t \in [f(t), 1] \end{cases}$

- Surjective:

Definition (Path Lifting). *Let $p : E \to B$ be a map and $f : X \to B$ be a continuous map. A lifting of f is a map $\tilde{f} : X \to E$ such that $p \circ \tilde{f} = f$. The following diagram represents the situation:*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Given $\gamma : I \to S^1, \tilde{\gamma} : I \to \mathbb{R}$ is a path lift of γ if $\exp \circ \tilde{\gamma} = \gamma$, where $\exp(t) = e^{2\pi i t}$.

Definition (Evenly Covered Neighbourhood). *Let $f : X \to Y$ be a continuous surjective map. An open set $U \subseteq Y$ is said to be evenly covered by f (or an evenly covered neighbourhood) if $f^{-1}(U) = \sqcup V_\alpha$ such that for each α, V_α is open in X and $f|_{V_\alpha} \to U$ is a homeomorphism.*

Lemma 1.2. *The following will help us establish surjectivity.*

1. *Given $\gamma : I \to S^1$, there exists $0 = t_0 < t_1 ... < t_k = 1$ such that $\gamma([t_i, t_{i+1}])$ lies in some evenly covered neighbourhood of $\exp$.*
2. *Given $\gamma : I \to S^1$, there exists a lift $\tilde{\gamma} : I \to \mathbb{R}$ which is unique if $\tilde{\gamma}(0) \in \exp^{-1}(\gamma(0))$ is fixed.*

Proof. We prove the statements in order.

1. Let $t \in I$, and consider $U_t \ni \gamma(t)$ such that U_t is evenly covered by exp. We can always find such a U_t because, observe that we can write, for $U = S^1 \backslash \{-e^{i\theta}\}, \exp^{-1}(U) = \mathbb{R} \backslash \exp^{-1}(e^{-i\theta}) = \mathbb{R} \backslash (\mathbb{Z} + \theta) = \sqcup_{k \in \mathbb{Z}} V_k$ for any θ.
   S^1 is a compact metric space, and $\{U_t\}$ will form an open cover of it. Therefore, by the Lebesgue number lemma, there will exist a $\delta > 0$ such that any set with diameter $< \delta$ will lie in some U_t.
   We can now simply choose a partition $\gamma([0, t_1], \gamma([t_1, t_2])..., \gamma([t_{k-1}, 1])$ such that each set has diameter $< \delta$; then, each will lie in some evenly covered neighbourhood.
2. Now, we know there exists $0 = t_0 < t_1 ... < t_k = 1$ such that $\gamma([t_i, t_{i+1}])$ lies in some evenly covered neighbourhood. So consider $\gamma|_{[t_0, t_1]} \subseteq U_0, \exp^{-1}(U_0) = \sqcup_{k \in \mathbb{Z}} V_k^0$. Taking $\tilde{\gamma}(0)$ as given, pick a k such that $\tilde{\gamma}(0) \in V_k^0$, and define $\tilde{\gamma} : [t_0, t_1] \to \mathbb{R}, s \mapsto (\exp|_{V_k^0})^{-1}(\gamma(s))$. (This is well-defined and continuous because the restriction of exp is a homeomorphism.)
   We can now use the same method to define $\tilde{\gamma}$ on the remaining intervals and use the pasting lemma to put together $\tilde{\gamma} : I \to \mathbb{R}$.

$\square$

Now, to show that Φ is surjective, we need to show that an arbitrary loop $\gamma : I \to S^1$ is homotopic to some γ_n; that is, $\exp(\tilde{\gamma}) \sim \exp(\tilde{n})$, where $\tilde{\gamma}, \tilde{n}$ are the unique path lifts of γ, γ_n. (Note that we set $\gamma(0) = \gamma(1) = 1, \tilde{\gamma}(0) = 0$.) Now this will follow if we show that $\tilde{\gamma}$ and $\tilde{n}$ are based homotopic as paths.
To see this, set $n = \tilde{\gamma}(1) - \tilde{\gamma}(0)$. (This will obviously be an integer.) Clearly, $\tilde{n}(t) = (\exp)^{-1} e^{2\pi i n t} = nt = t\tilde{\gamma}(1) + (1-t)\tilde{\gamma}(0)$. This will be path homotopic to $\tilde{\gamma}$ rather trivially, thanks to the fact that $\mathbb{R}$ is convex: $h(s,t) = s\tilde{\gamma}(t) + (1-s)\tilde{n}(t)$.

- Injective:

Definition (Path homotopy lifting). *Let $\gamma, \gamma' : I \to S^1$ be homotopic via $h : I \times I \to S^1$, and $\tilde{\gamma} : I \to \mathbb{R}$ be a path lifting of γ. Then, the unique $\tilde{h} : I \times I \to \mathbb{R}$ is called the path homotopy lift of h, and is such that $\tilde{h}(-,0) = \tilde{\gamma}, \exp \circ \tilde{h} = h$.*

We shall defer the proof of the existence and uniqueness of $\tilde{h}$ to the next section, where a more general version of this statement is proven.

To show that Φ is injective, we can show its kernel is 0, which is that $[\gamma_n] = [\gamma_0] \implies n = 0$; in other words, $\gamma_n \sim \gamma_0 \implies n = 0$.
Now, let $h : I \times I \to S^1$ be a homotopy of based loops between $\gamma_0, \gamma_n : I \to S^1$ with $h(0,-) = h(1,-) = 1$. By homotopy lifting, we have a unique $\tilde{h} : I \times I \to \mathbb{R}$ such that $\tilde{h}(-,0) = \tilde{\gamma}_n$. By the uniqueness of $\tilde{h}$, this will also fix $\tilde{h}(-,1) = \tilde{\gamma}_0$.
Now, we have fixed $\tilde{h}(1,1) = 0$, and $\tilde{h}(1,-) = \exp^{-1}(h(1,-)) = \exp^{-1}(1)$ must be an integer. Since $\tilde{h}(1,t)$ is a continuous function and I is path connected, we have $\tilde{h}(1,-) = 0$ throughout.
But also, $\tilde{h}(1,0) = n \implies n = 0$, and we are done.

$\square$

The Homotopy Lifting Property

Definition (Covering). *A map $\pi : E \to B$ is called a covering map, and E a covering space of B, if for every $x \in B$ there is an evenly covered neighbourhood U such that $x \in U$.*

Definition (Homotopy lifting). *We say that $\pi : E \to B$ has the homotopy lifting property if the following diagram admits an $\tilde{h}$ such that it commutes for all $Y, h, \tilde{h}_0$:*

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, hook] & E \arrow[d, "\pi"] \\
Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}$$

$\tilde{h}$ is called the homotopy lift of h.

Remark. If we put $Y = \{*\}$, we get back path lifting; if $Y = I$, we get back path homotopy lifting.

Theorem 2.3. *Covering maps have unique homotopy liftings for each Y.*

Proof. Let $\pi : E \to B$ be a covering map and $h : Y \times I \to B$ be a homotopy. We need to show that there exists a lift $\tilde{h} : Y \times I \to E$ of h, and further, that if $\tilde{h}(-, 0)$ is given, $\tilde{h}$ is unique.

Consider, to begin with, for a given homotopy $h : Y \times I \to B, h_y : \{y\} \times I \to B$. This is just a path, and so we know it can be lifted to a unique path $\tilde{h}_y$ in E (given the starting point).

We claim that $\tilde{h} := \tilde{h}_y(t) \forall y \in Y, t \in I$ is the homotopy lift. That such an $\tilde{h}$ would make the above diagram commute follows from the fact that each path lift would. So, what remains to be checked is:

- Uniqueness: Since each path lift is unique, so will the whole thing be, and this is done.
- Continuity:

$\square$

Applications

The Homotopy Lifting Property

Theorem 2.4. *Let $\pi : \tilde{X} \to X$ be a covering map and X be path connected. Then, all fibers (inverse image of singletons) will have the same cardinality.*

Proof. Let $x_1, x_2, \in X$, and γ be a path from x_1 to x_2. Define a map $\varphi_\gamma : \pi^{-1}(x_1) \to \pi^{-1}(x_2), \tilde{x} \mapsto \tilde{\gamma}_{\tilde{x}}(1)$, where $\tilde{\gamma}$ is the path lift of γ which starts at $\tilde{x}$. Uniqueness of the path lift forces this map to be injective. The analogous map defined on $\bar{\gamma}$ will be injective for the same reason.
Now, $\varphi_{\bar{\gamma}}(\varphi_\gamma(\tilde{x}_1)) = \varphi_{\bar{\gamma}}(\tilde{\gamma}_{\tilde{x}_1}(1)) = \varphi_{\bar{\gamma}}(\tilde{x}_2) = \tilde{\bar{\gamma}}_{\tilde{x}_2}(1) = \tilde{x}_1 \implies \varphi_{\bar{\gamma}} \circ \varphi_\gamma = Id_{\pi^{-1}(x_1)}$. Similarly, $\varphi_\gamma \circ \varphi_{\bar{\gamma}=Id_{\pi^{-1}(x_2)}}$.
Thus, the maps are isomorphisms, proving the required. $\square$

Theorem 2.5. *Let $\pi_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ be the map induced by the covering. Then, π_* is injective.*

Proof. We shall show that $\ker(\pi_*) = 0 \iff [\pi(\gamma)] = 0 \implies [\gamma] = 0$, where γ is a based loop in the covering space.
Let $h : S^1 \times I \to X$ be a homotopy of based loops between $\pi(\gamma)$ and C_{x_0}. We would like to find another homotopy of based loops $\tilde{h}$ between γ and $C_{\tilde{x}_0}$ using h.
We can apply the homotopy lifting property with $Y = S^1, h$ as above and $\tilde{h}(-, 0) = \gamma$ to conclude the existence of a homotopy lift $\tilde{h}$ which will further satisfy $\pi \circ \tilde{h}(-, 1) = h(-, 1) = C_{x_0} \implies \tilde{h}(-, 1) = \pi^{-1}(x_0)$. Now, note that, since π is continuous and $\{x_0\}$ is a connected set, its preimage will also be connected.
This forces it to be a constant, because fibers are discrete under the subspace topology: For let U be an openly covered neighbourhood around x_0, so that $\pi^{-1}(U)$ is the disjoint union of open sets U_i homeomorphic to U. $U_i \cap \pi^{-1}(x_0)$ is forced to be a singleton for each i simply because $\pi|_{U_i}$ is a homeomorphism and so injective.
Similarly, $\tilde{h}(1, -)$ is constant. But we already know that $\tilde{h}(1, 0) = \tilde{x}_0$. Therefore, $\tilde{h}(-, 1) = C_{\tilde{x}_0}$. $\square$

Theorem 2.6 (Degree of a cover)**.** *The size of a fiber equals the index of $\pi_1(\tilde{X}, \tilde{x}_0)$ in $\pi_1(X, x_0)$.*

Proof. Not done in class. □

The Fundamental Group of the Circle

Theorem 2.7 (Brouwer's Fixed Point Theorem)**.** *Any continuous $h : \mathbb{D}^2 \to \mathbb{D}^2$ has a fixed point.*

Proof. Suppose no fixed point exists. Conveniently center the disk at the origin and consider $r : \mathbb{D}^2 \to S^1, x \mapsto (1-t_0)h(x) + t_0 x$, where t_0 is such that $||r(x)|| = 1$. What this map 'does' is extend the line joining $h(x), x$ to the boundary of the disk.
Observe that r is a deformation retract: Continuity follows trivially from that of h, and $r|_{\partial\mathbb{D}^2} = Id_{S^1}$. This itself is a contradiction, because unfortunately, there can be no retract of $\mathbb{D}^2$ to S^1. The argument is as follows:
Let i be the inclusion map $S^1 \hookrightarrow \mathbb{D}^2$. We will have induced maps $i_* : \pi_1(S^1) \to \pi_1(\mathbb{D}^2), r_* : \pi_1(\mathbb{D}^2) \to \pi_1(S^1)$; the induced map takes (the equivalence class of) a loop in the domain parent space to the (equivalence class of) the loop given by the image of the original loop under the original map.
Now, $r \circ i = Id_{S^1} \implies (r \circ i)_* = Id_{\pi_1(S^1)}$. But also, since $\pi_1(\mathbb{D}^2) = 0, i_*$ must be the trivial map, so that $(r \circ i)_* = r_* \circ i_* = 0$, which brings out the required contradiction.
□

Theorem 2.8 (Fundamental Theorem of Algebra)**.** *Every non-constant polynomial in $\mathbb{C}[x]$ has a root in $\mathbb{C}$.*

Proof. Suppose $p(z) = z^n + a_1 z^{n-1} + ...a_{n-1}z + a_n$ is a polynomial in $\mathbb{C}$ with no roots. We will show that this forces $n = 0$.
Consider $H_r(s) : I \to S^1, H_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|}, r \geq 0$. For each value of t, this is a loop on S^1 based at 1 ($H_r(1) = 1 \forall r$). Furthermore, $H_0(s) = 1$, so that H_0 is the constant loop. But note that H_r will be homotopic to H_0 for any r (simply define $h(s,t) : I \times I \to S^1, h(s,t) = H_{tr}$). Thus, $[H_r] = 0 \in \pi_1(S^1)$.
Now, choose $r > \max\{1, |a_1| + ... + |a_n|\}$. Then, for $|z| = r, |z|^n = r \cdot r^{n-1} > (|a_1| + ...|a_n|)|z^{n-1}|) \geq |a_1 z^{n-1} + ...a_n z^{n-1}| \geq |a_1 z^{n-1} + ...a_n|$. From this inequality, we may conclude that the polynomial $p_t(z) = z^n + t(a_1 z^{n-1} + ...a_n)$ has no roots on the circle $|z| = r$ for $t \in I$.
Now, for this fixed r, define the homotopy $F(s,t) = \frac{p_{1-t}(re^{2\pi is})/p_{1-t}(r)}{|p_{1-t}(re^{2\pi is})/p_{1-t}(r)|}$. $F(1,t) = 1, F(s,0) = H_r(s) \sim \varphi_0$, and $F(s,1) = e^{2\pi ins} = \varphi_n \implies \varphi_0 \sim \varphi_n \implies n = 0$. □

Theorem 2.9 (Borsuk-Ulam Theorem)**.** *If $f : S^2 \to \mathbb{R}^2$ is a continuous map, there exist a pair of points $\{x, -x\} \subset S^2$ such that $f(x) = f(-x)$.*

Proof. Suppose no such pair exists for some continuous function f, and consider $g : S^2 \to S^1, x \mapsto \frac{f(x)-f(-x)}{|f(x)-f(-x)|}, \eta : I \to S^2, s \mapsto (\cos 2\pi s, \sin 2\pi s, 0)$, and finally, $\gamma := g \circ \eta : I \to S^1$. It is easy to see that γ is continuous, and since $\gamma(0) = \gamma(1)$, we see that γ is a loop based at

$\frac{f(e_1)-f(-e_1)}{|f(e_1)-f(-e_1)|} = x_0$, where $e_1 = (1,0,0)$.

- γ is not nullhomotopic: Lift γ to $\tilde{\gamma} : I \to \mathbb{R}$. Since $\gamma(s+\frac{1}{2}) = -\gamma(s), s \in [0, \frac{1}{2}]$, we also have $\tilde{\gamma}(s+\frac{1}{2}) = -\tilde{\gamma}(s) + \frac{q}{2}$, where q is an odd integer. [Argument incomplete.]
- γ is nullhomotopic: First, note that η is nullhomotopic, via $h(s,t) = ((1-t^2)\cos 2\pi s, (1-t^2)\sin 2\pi s, t)$. Then, $\gamma \sim C_{x_0}$ via $g \circ h$.

This brings out the required contradiction, and we conclude that such a pair of points must exist. $\square$

Interlude: Connectedness

Definition. *A topological space X is **connected** if it cannot be written as the union of two disjoint non-empty open sets.*

Definition. *A topological space X is **path connected** if for any $x, y \in X$, there exists a continuous $\gamma : I \to X$ such that $\gamma(0) = x, \gamma(1) = y$.*

Definition. *A topological space X is **simply connected** if it is path connected and it has trivial fundamental group.*

Definition. *A topological space X is **locally path connected** if every $x \in X$ has a path connected open neighbourhood.*

Definition. *A topological space X is **locally simply connected** if every $x \in X$ has a neighbourhood U which is simply connected, that is, every loop in U is nullhomotopic in U.*

Definition. *A topological space X is **semilocally simply connected** if every $x \in X$ has a neighbourhood U such that every loop in U is nullhomotopic in X.*

Covering Space Theory

The Universal Cover

Let X be a path-connected topological space. We want to find a cover $\tilde{X}$ which is simply connected. Such a cover shall be called a *universal cover* of X for reasons which will become clear.

First of all, suppose such a cover exists. Then, X must be semilocally simply connected: This amounts to saying that every $x \in X$ has a neighbourhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is the zero map.
Choose an evenly covered neighbourhood and let $\pi^{-1}(x_0) \ni \tilde{x_0} \in \tilde{U}$ such that $\pi|_{\tilde{U}} \cong U$.
We have the following commutative diagram:

$$\begin{tikzcd} \tilde{U} \arrow[r, hook, "\tilde{i}"] \arrow[d, "\pi"] & \tilde{X} \arrow[d, "\pi"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd}$$

This will induce maps between the fundamental groups:

$$\begin{tikzcd} \pi_1(\tilde{U}, \tilde{x}) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & \pi_1(\tilde{X}, \tilde{x}) \arrow[d, "\pi_*"] \\ \pi_1(U, x) \arrow[r, hook, "i_*"] & \pi_1(X, x) \end{tikzcd}$$

But since $\tilde{X}$ is simply connected, its fundamental group will be trivial. This forces i_* to be the zero map.

Now, suppose γ, η are two paths in X based at x_0, and they are based homotopic via h. By the homotopy lifting property, $\tilde{\gamma}, \tilde{\eta}$ are based paths in $\tilde{X}$ which are *based* homotopic via $\tilde{h}$. They obviously agree at the starting point; we will also have $\tilde{\gamma}(1) = \tilde{\eta}(1)$, because both of them lie in $p^{-1} \circ (h(-, 1)) = \tilde{h}(-, 1)$, which is a connected discrete set (connected by virtue of being the continuous pre-image of a connected set, and discrete by virtue of being the pre-image of a fiber).

It is obvious that if two paths are based homotopic in the covering space, they will be based homotopic in the base space.

Therefore, $\gamma \sim_b \eta \iff \tilde{\gamma} \sim_b \tilde{\eta}$.

We now have enough heuristic constraints to furnish a final construction of the universal cover. Observe how *any* $\tilde{x} \in \tilde{X}$ is associated with an equivalence class of paths $[\gamma]$, where γ is a path in X and two paths are related if they are based homotopic as paths: Given a γ, lift it and see the lift's endpoint. This will remain the same for all elements of the class. Conversely, for any element in the cover, we will obviously have a path from the basepoint to it; projecting this down will give us the path whose equivalence class will identify the point (due to path connectedness of X).

Definition. *The universal cover of a path-connected & semilocally simply connected space X is $\tilde{X} = \mathcal{P}_{x_0}X/\sim$, where $f \sim g \iff f(1) = g(1)$ and the two are based homotopic as paths. The topology on it is the one generated by the basis consisting of sets of the form $U_\gamma = \{[\gamma * \alpha] \in \tilde{X} | \gamma \in \mathcal{P}_{x_0}X, \gamma(1) \in U, U \text{ is open and } \alpha \text{ is a path in } U\}$.*

Theorem 3.1. *$\tilde{X}$ is simply connected, and* $\mathrm{ev} : \tilde{X} \to X, [\gamma] \mapsto \gamma(1)$ *is a covering map.*

Intermediate Covers: Bottom-up

Consider the space $X_H := \mathcal{P}_{x_0}X/ \sim_H$, where $\gamma \sim_H \eta \iff \gamma(1) = \eta(1)$ and $[\gamma\bar{\eta}] \in H \leq \pi_1(X, x_0)$.

Notice how we get back the universal cover for $H = \{e\}$, and the base space for $H = \pi_1(X, x_0)$ (for in the latter case, the equivalence constraint merely says that two paths are identified simply if they form a loop without requiring said loop to be in any particular subgroup of the fundamental group—thereby associating any $x \in X$ uniquely with the equivalence class of paths which end at it).

So:

- $\widetilde{X} = \widetilde{X}/\{e\}$
- $X_H = \widetilde{X}/H$
- $X = \widetilde{X}/\pi_1(X, x_0)$

We have a kind of 'tower' of topological spaces ordered by the size of the subgroup quotiented by. The universal cover justifies its name by being that which has to be quotiented by a different subgroup each time for a new cover to be produced.

All this suggests the following theorem:

Theorem 3.2. *X_H is a connected cover of X, and $\pi_1(X_H) \cong H$.*

Definition. *Two covers are called equivalent if there exists a homeomorphism φ such that the following diagram commutes:*

$$\begin{tikzcd}
(\hat{X}, \hat{x_0}) \arrow[rr, "\varphi"] \arrow[dr, "\hat{p}"'] & & (X', x_0') \arrow[dl, "p'"] \\
& (X, x_0) &
\end{tikzcd}$$

If, in particular, we set $X' = \hat{X}$, the homeomorphisms form a group under the composition operation. This is called the group of deck transformations of the covering space, denoted by $G(\hat{X})$.

Definition. *A cover is called* ***regular*** *if there exists a deck transformation between any two pre-images of the base point.*
A cover is called ***normal*** *if it is connected and its image under p_* in the fundamental group of the base space is a normal subgroup.*

We shall now rattle off a number of results.

Theorem 3.3. *Let (X, x_0) be a based space, $(\tilde{X}, \tilde{x}_0)$ a cover of it with map p, and Y be some path-connected and locally path-connected based space. The following hold:*

1. *For any continuous $f : (Y, y_0) \to (X, x_0)$, a lift of f exists $\iff$ $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*
2. *$(\tilde{X}, \tilde{x}_0)$ is a connected regular cover $\iff$ It is a normal cover.*
3. *$G(X_H) \cong N(H)/H$.*

Proof. This will be long and painful.

1. We prove each implication in turn.
   $\implies$: This one is easy. Suppose a lift $\tilde{f}$ exists. Then, $p \circ \tilde{f} = f \implies p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \supseteq p_* \circ \tilde{f}_*(\pi_1(Y, y_0)) = f_*(\pi_1(Y, y_0)) \implies f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

   $\impliedby$: Given that the condition holds, we need to define a lift of f.
   Let $\tilde{f}(y) := \widetilde{f(y)}$, the endpoint of the path lift of $f(\gamma)$ initialized at an $\tilde{x}_0 \in p^{-1}(x_0)$, where γ is a path from y_0 to y. We claim that this is a lift of f.

2. The following lemma will help us out.

   Lemma 3.4. *Two connected based covers are equivalent $\iff$ Their induced images are equal.*

   $\implies$: We begin by showing that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H_0$ and $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H_1$ are conjugates of each other. Consider a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then, $p(\tilde{\gamma}) = g \in \pi_1(X, x_0)$. Now, for any loop $\tilde{f}$ based at $\tilde{x}_0, \tilde{\gamma}^{-1} * \tilde{f} * \tilde{\gamma}$ will be a loop based at $\tilde{x}_1 \implies g^{-1} * H_0 * g \subseteq H_1$. Similarly, $H_0 = g * H_1 * g^{-1}$, thus proving that the two are conjugates of one another.

   Now, let $(\tilde{X}, \tilde{x}_0)$ be a connected regular cover, and let $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$. We want to show that $g^{-1} * H * g = H$ for all $g \in \pi_1(X, x_0)$. Let $g = [\gamma]$, and consider the endpoint $\tilde{x}_1 \in p^{-1}(x_0)$ of the lift of γ initialized at $\tilde{x}_0$. We know by the above argument that $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = g^{-1} * H * g$. But also, by assumption of their equivalence, $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = H$. This proves that H is normal.

   $\impliedby$: Next, we suppose $(\tilde{X}, \tilde{x}_0)$ is a normal cover. Then, $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = g^{-1} * H * g$ for all $g \equiv [\gamma] \in \pi_1(X, x_0)$ Consider any $\tilde{x}_1 \in p^{-1}(x_0)$. By the above, $p_*((\tilde{X}, \tilde{x}_1)) = H\prime$ is a conjugate of H. But since H is normal, $H = H'$; and so by the lemma, $(\tilde{X}, \tilde{x}_1)$ is equivalent to $(\tilde{X}, \tilde{x}_0)$. Since this is true for all the pre-images of x_0, this proves regularity.

3. Define $\Phi : N(H) \to G(X_H), [\gamma] \mapsto \varphi_{\tilde{\gamma}}$, the deck transformation taking $\tilde{\gamma}(1) = \tilde{x}_1$ to $\tilde{x}_0$. We claim that this is a surjective group homomorphism with kernel H; the result will then follow from the first isomorphism theorem.

□

Corollary 3.4.1. *The following bijective correspondences hold:*

- *Subgroups $\longleftrightarrow$ Isomorphism classes of connected based covers*
- *Subgroups up to conjugation $\longleftrightarrow$ Isomorphism classes of connected covers*

Theorem 3.5. *Let X_H be a normal cover of X. Then, $X_H/G(X_H) \cong X$.*

Intermediate Covers: Top-down

Lemma 3.6. *Let X be a topological space, and G be a group which acts properly discontinuously on it (that is, for all $x \in X$, there exists an open set $U \ni x$ such that $g \cdot U \cap U = \emptyset$ for all $g \neq e$).*

Given a (reasonable) based space (X, x_0), a discrete group G, and a group homomorphism $\rho : \pi_1(X, x_0) \to G$ (which induces a natural (left) group action of π_1 on G, $[\gamma] \cdot g := \rho([\gamma])g$), we wish to construct a covering $Y \to Y/G \cong X$, where G will act properly discontinuously on Y.

Theorem 3.7. *Let $X_\rho := (\tilde{X} \times G)/\sim$, where $(\tilde{x}, g) \sim (\tilde{y}, g') \iff \exists[\gamma] \in \pi_1(X, x_0)$ such that $[\gamma] \cdot g = g', [\gamma * \alpha]_{\tilde{X}} = \tilde{x}$ (where $\tilde{x} = [\alpha]_{\tilde{X}}$, if we recall the universal cover's construction).*

Then, the (right) action of G on X_ρ given by $(\tilde{x}, g) \cdot g_1 \mapsto (\tilde{x}, gg_1)$ is properly discontinuous. Furthermore, $X_\rho/G \cong X$.

Lemma 3.8. *ρ is surjective $\iff$ X_ρ is connected.*

Lemma 3.9. *Let (X, x_0) be a path-connected space with universal cover. Then, for any group G, the following sets stand in bijective correspondence:*

$$\mathrm{Hom}(\pi_1(X, x_0), G) \leftrightarrow \{\textit{Based } G\textit{-regular covers of } (X, x_0)\}/G - \textit{equivalence}$$

Here, two covers are G-equivalent if they are equivalent as covers, and if the homeomorphism satisfies $\varphi(g \cdot z) = g \cdot \varphi(z)$.

Seifert-Van Kampen theorem

Definition. *The amalgamated product of two free groups, denoted by $F_1 *_H F_2$ (where $H \leq F_1, H \leq F_2$), is their free product quotiented by the normal subgroup generated by$\{\varphi_1(h)\varphi_2(h^{-1}), h \in H\}$, where φ_1, φ_2 are the inclusion maps of H into F_1, F_2 respectively.*

Theorem 3.10. *Let $X = U \cup V$ be such that U, V are non-empty open sets, $X, U, V, U \cap V$ all have a universal cover, and $U, V, U \cap V \ni x_0$ are all path-connected. Then, there exists an isomorphism $\varphi : \pi_1(U, X_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$.*

Theorem 3.11. *$\mathcal{F}_2$ is isomorphic to a subgroup of $\mathcal{F}_3$. $\mathcal{F}_3$ is isomorphic to a subgroup of $\mathcal{F}_2$. Also, $\mathcal{F}_2, \mathcal{F}_3$ are not isomorphic.*

Proof. There is an obvious inclusion map $\mathcal{F}_2 \to \mathcal{F}_3$. For the converse, we shall construct two topological spaces with fundamental group $\mathcal{F}_2, \mathcal{F}_3$, such that the latter is a cover of the former. The first result will then follow from theorem 2.5.
We will use the fact that the k-bouquet of circles has fundamental group $\mathcal{F}_k$.

-  covers :
- $\pi_1($$)= \mathcal{F}_3$: It suffices to show homotopy equivalence of the same with the 3-bouquet of circles.

$\square$

Notes on Homology & Cohomology

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Simplicial Homology

Definition (n-simplex). *An n-simplex is the set of points $\Delta^n = \{\sum_{i=1}^n t_i v_i | \sum_{i=1}^n t_i = 1, t_i \geq 0\}$, such that $v_1 - v_0, ... v_n - v_0$ are linearly independent vectors in $\mathbb{R}^{n+1}$.*

- *The coefficients t_i are the **barycentric coordinates** of the point in Δ^n*
- *The points $v_i \in \mathbb{R}^{n+1}$ are the **vertices** of the simplex. The set of vertices will be given an ordering.*
- *The standard n-simplex is the one where the vertices are the unit vectors along the coordinate axes.*
- *A **face** of a simplex is the subsimplex with vertices as any nonempty subset of the original one.*
- *The union of all the faces of Δ^n is called its **boundary** and denoted by $\partial\Delta^n$.*
- *The **open simplex** $\Delta^n - \partial\Delta^n$ is its interior.*

Definition (Δ-complex). *A $\Delta-$complex structure is a collection of maps $\sigma_\alpha : \Delta^n \to X$, with n depending on α, such that:*

1. *The restriction of each map to the interior of the simplex is injective, and each point in X is the image of exactly one such restriction*
2. *The restriction of each map to a face is another map $\sigma_\beta : \Delta^{n-1} \to X$ in the collection*
3. *$A \subseteq X$ is open $\iff$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.*

Discussion. A simplex is intended to be a generalization of the triangle. We were interested in triangles because any polygon can be broken down into triangles with edges identified; we were interested in polygons, in turn because any surface can be represented as a polygon with appropriately identified edges. Through triangles and their identifications alone, many two-dimensional spaces can be constructed.
Based on the above definitions, one can see how a torus, the projective plane, and the Klein bottle now all fall under the rubric of a Δ-complex based off the natural 2-simplex. Another example which I mention because it showed up in the course was the dunce hat, wherein we take a 2-simplex and identify all its faces.
A Δ-complex can also be thought of as built in the following way: Given Δ^n_α be a collection of disjoint simplices and $\mathscr{F}_i$ a collection of faces taken from all of them (such that the simplices may have varying dimension but the faces have the same dimension), a Δ-complex X is the disjoint union of the simplices quotiented by an identification of the faces $\mathscr{F}_i$ through the canonical homeomorphism between them.

Remark. A word on orientation: The orientation of the edges in a $\Delta-$complex will invariably be inherited from the ordering of the vertices in the parent simplices, which implies the imposition of certain natural conditions on them. For example, no 2-simplex can have its edges oriented cyclically.

Definition. *$\Delta_n(X)$ is the free abelian group with basis the open n-simplices of X. Its elements, called **n-chains**, may be written without ambiguity as $\sum_\alpha n_\alpha \sigma_\alpha$.*

Definition. *The boundary of the n-simplex $[v_0, ..., v_n]$ is $\sum_i (-1)^i [v_0, ..., \hat{v}_i, ..., v_n]$, where the hat indicates deletion. The boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ is defined by $\sigma_\alpha \mapsto \sum_i (-1)^i \sigma_\alpha | [v_0, ..., \hat{v}_i, ..., v_n]$.*

Lemma 1.1. $\partial_n \partial_{n+1} = 0$ ($\iff \mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$).

Definition. *The n^{th} simplicial homology group of a Δ-complex X, $H^\Delta_n(X)$, is defined as the quotient $\mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$.*

*Elements of $\mathrm{Ker}(\partial)$ are called **cycles**, and elements of $\mathrm{Im}(\partial)$ are called **boundaries**.*

Discussion. The idea with the boundary homomorphism is to orient the faces of the simplex coherently.
$\Delta-$complexes are the more general version of simplicial complexes, which is what most texts other than Hatcher seem to be using. The former are advantegeous insofar as spaces can be triangulated more quickly with them; the tradeoff is that, in a simplicial complex, given a set of ordered vertices, the simplex it belongs to is uniquely determined.
More formally, given a $\sigma : \Delta^n \to X$, a simplicial complex cannot have a $\tau : \Delta^n \to X$ such that $\sigma(v_i) = \tau(v_i)$ for all the vertices v_i of Δ^n.

Example. Let us compute the simplicial homology groups of two basic figures.

- The circle: This can be constructed as a Δ-complex by identifying the two vertices of a 1-simplex. Overall, then, it will have one 0-simplex (v), one 1-simplex (e), and nothing of higher dimension. Therefore, $\Delta_0(X) = \Delta_1(X) = \mathbb{Z}$, and $\Delta_n(X) = 0$ for $n \geq 2$.
  This makes it immediate that the boundary map ∂_n will have trivial kernel for $n \neq 0, 1$, and kernel $\mathbb{Z}$ for $n = 0$. It will also have trivial image for all $n \neq 1$.
  It remains to compute $\partial_1 : \mathbb{Z} \to \mathbb{Z}$. But now $\partial_1(e) = v - v = 0$, since both the vertices of the 1-simplex are identified.
  We now know that the first two homology groups are $\mathbb{Z}$, and all the others are trivial.

- The torus: One can read off from the figure near the beginning of the section that the $\Delta-$complex triangulation by way of the polygonal representation will have you end up with one 0-simplex, three 1-simplexes, and two 2-simplexes. So, we will be dealing with $C_0 = \mathbb{Z}, C_1 = \mathbb{Z}^3, C_2 = \mathbb{Z}^2$.
  Again, for ∂_0, the kernel is obviously $\mathbb{Z}$ and image trivial. It is also clear that the homology groups of order ≥ 3 are all trivial. What remains to be seen are the boundary homomorphisms corresponding to $n = 1, 2$.
  Once again, since all the vertices are identified, we will have $\partial_1 = 0$, since it will vanish on each of the three generators a, b, c of C_1. This determines its image and kernel.
  The image of each of the 2-simplexes U, L under ∂_2 will be $a+b-c$. Thus, $\mathrm{Im}(\partial_2)$ is the infinite cyclic group generated by $a+b-c$. Finally, $\partial_2(pU+qL) = (p+q)(a+b-c) = 0 \implies p = -q$. Thus, $\mathrm{Ker}(\partial_2)$ is the infinite cyclic group generated by $U - L$.
  Having computed all the relevant kernels and images, arriving at the final homology groups is quick work.

Singular homology

Definition. *A **singular n-simplex** on a topological space X is a continuous map $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex.*

Remark. The definition of a singular homology group is nearly identical to that of the simplicial homology group. The only difference is that the basis of the free abelian groups in the chain complex is the set of singular n-simplices in X, rather than the set of open simplices.
Note that we could only speak of the simplicial homology groups of a space through its Δ-complex structure. No such thing seems to be required for the singular homology groups of a space.

Theorem 1.2. *If a space X can be decomposed into path components $\{X_\alpha\}, H_n(X)$ is isomorphic to $\bigoplus_\alpha H_n(X_\alpha)$.*

Proof. Since a singular n-simplex is a continuous map and an n-simplex itself is path-connected, its image will be path-connected. So, the group $C_n(X)$ will split into a direct sum $C_n(X_\alpha)$, and the boundary maps will preserve this split in both image and kernel. Hence, the homology groups will also split. $\square$

Theorem 1.3. *If X is non-empty and path-connected, $H_0(X) = \mathbb{Z}$.*

Proof. $H_0(X) = C_0(X)/\text{Im}(\partial_1)$. Define $\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum_i n_i\sigma_i \mapsto \sum_i n_i$. This is obviously surjective. If we show that its kernel equals $\text{Im}(\partial_1)$, we will have $H_0(X) \cong \mathbb{Z}$ by the first isomorphism theorem.

- $\text{Im}(\partial_1) \subseteq \text{Ker}(\epsilon)$: $\epsilon \circ \partial_1(\sigma) = \epsilon(\sigma|[v_1, \hat{v_0}] - \sigma|[\hat{v_1}, v_0]) = 1 - 1 = 0$, where $\sigma : \Delta^1 \to X$ is a singular 1-simplex.
- $\text{Im}(\partial_1) \supseteq \text{Ker}(\epsilon)$: Suppose $\epsilon(\sum_i n_i\sigma_i) = 0 \implies \sum_i n_i = 0$. We need to show $\sum_i n_i\sigma_i$ is the boundary of some element in $C_1(X)$.
  The σ_i are maps from 0-simplices and so map to points of X. Since we assumed path-connectedness, choose a path $\tau_i : I \to X$ from a basepoint $x_0 \in X$ to $\sigma_i(v_0)$. Further, let σ_0 be the singular 0-simplex which maps to x_0.
  We claim that $\sum_i n_i\sigma_i$ is the boundary of $\sum_i n_i\tau_i \in C_1(X)$.
  Firstly, we can consider each τ_i to be a singular 1-simplex from $[v_0, v_1]$ to X with some resizing. Next, note that $\partial_1(\tau_i) = \sigma_i - \sigma_0$.
  Therefore, $\partial(\sum_i n_i\tau_i) = \sum_i n_i\sigma_i - \sum_i n_i\sigma_0 = \sum_i n_i\sigma_i$, since $\sum_i n_i = 0$. This completes the proof.

□

Theorem 1.4. *If X is a point, $H_n(X) = 0$ for $n > 0$.*

Proof. For the below, suppose $n \neq 0$.
There is a unique singular n-simplex σ_n for each n, and $\partial_n(\sigma_n) = 0$ if n is odd and σ_{n-1} if even. So, depending on n, we will either have $H_n(X) = \mathbb{Z}/\mathbb{Z}$ or $0/\mathbb{Z}$. In either case, it is trivial. □

Homotopy invariance

Definition. *For any (continuous) map $f : X \to Y$, there is an induced homomorphism $f_\# : C_n(X) \to C_n(Y)$, $f_\#(\sum n_i\sigma_i) = \sum n_i f_\#(\sigma_i)$, where $f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y \in C_n(Y)$.*

Lemma 1.5. $f_\#\partial = \partial f_\#$

Proof. Just compute $f_\#\partial$. This is almost immediate from definitions. □

Remark. We say that the $f_\#$s define a **chain map** from the singular chain complex of X to that of Y.

Lemma 1.6. *A chain map between chain complexes $f_\#$ induces homomorphisms f_* between the homology groups of the complexes.*

Proof. If $\alpha \in \ker(\partial_n)$ of X, then $\partial\alpha = 0 \implies f_\#(\partial\alpha) = \partial(f_\#\alpha) = 0 \implies f_\#(\alpha) \in \ker(\partial_n)$ of Y. Thus, $f_\#$ maps cycles to cycles. Similarly, it maps boundaries to boundaries. This allow us to define f_* in the obvious manner. □

Definition. *For $P : C_n(X) \to C_{n+1}(Y), \partial P + P\partial = g_\# - f_\# \iff P$ is a* ***chain homotopy*** *between the chain maps.*

Theorem 1.7. *If two maps $f, g : X \to Y$ are homotopic, they induce the same homomorphism $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. Given a homotopy $F : X \times I \to Y$ from f to g, define the **prism operators** $P : C_n(X) \to C_{n+1}(Y)$ as $P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times \mathbf{1})|[v_0, ..., v_n]$.
If we show P is a chain homotopy, we are done. For then, if $\alpha \in C_n(x)$ is a cycle, $g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) \implies (g_\# - f_\#)(\alpha)$ is a boundary in $C_n(Y)$. This implies the equality of f_*, g_* on the homology class of α; and this is true of each cycle α, so that we are done. $\square$

Corollary 1.7.1. *If $f : X \to Y$ is a homotopy equivalence, $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Corollary 1.7.2. *Chain-homotopic chain maps induce the same homomorphism on homology.*

Relative homology

Definition. *Given a space X and a subspace $A \subseteq X$, we denote by $C_n(X, A)$ the quotient group $C_n(X)/C_n(A)$.*

Discussion. We will have a sequence of boundary maps

$$... \to C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to ...$$

These will form another chain complex. We call its homology groups the **relative homology groups** $H_n(X, A)$.

- Elements of $H_n(X, A)$ will be such that $\alpha \in C_n(X), \partial\alpha \in C_{n-1}(A)$, since they must be in the kernel of the *quotient* boundary map. We call them **relative cycles**.
- Trivial elements of $H_n(X, A)$ will be such that they are in the image of the previous quotient boundary map, i.e., $\alpha = \partial\beta + \gamma, \beta \in C_n(X), \gamma \in C_n(A)$. We call them **relative boundaries**.

Theorem 1.8. *There is a long exact sequence of homology groups:*

$$... \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to ... \to H_0(X, A) \to 0$$

where i_*, j_* are the maps induced by the inclusion and the quotient respectively, and ∂ will be described in the proof.

Proof. First, note that we have a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X,A) & \longrightarrow & 0 \\ & & \downarrow{\partial} & & \downarrow{\partial} & & \downarrow{\partial} & & \\ 0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{j} & C_{n-1}(X,A) & \longrightarrow & 0 \end{array}$$

$\partial : C_n(X,A) \to C_{n-1}(X,A)$ is defined as the map which make the diagram commute. i, j will both obviously be chain maps here, so that everything commutes. We can write this in a condensed manner as $0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{j} C(X,A) \to 0$.
Next, note that, by virtue of being chain maps, i, j will induce maps i_*, j_* down to the homology groups.

Now, consider an element $[c] \in H_{n+1}(X,A), c \in C_{n+1}(X,A)$. This will be such that $\partial(c) = 0$. Also, by exactness, c will be in the image of $j : C_{n+1}(X) \to C_{n+1}(X,A)$ for some $x \in C_{n+1}(X)$. Since everything commutes, $j(\partial x) = \partial j(x) = \partial 0 = 0$, where we are now referring to $\partial : C_{n+1}(X) \to C_n(X)$. Therefore, $(\partial x) \in \mathrm{Ker}(j) = \mathrm{Im}(i), i : C_n(A) \to C_n(X)$. We write $\partial(x) = i(a), a \in C_n(A)$.
Lastly, $i(\partial a) = \partial i(a) = \partial\partial(x) = 0(\partial^2 = 0) \implies \partial(a) = 0$, since i is injective.

Therefore, we define $\partial : H_n(X,A) \to H_{n-1}(A), [c] \mapsto [a]$, where $[a]$ is located by the description above. Before doing anything else, we need to justify why this is well-defined:

- Since i is injective, $\partial(x)$ uniquely determines a.
- Had we chosen another $x', j(x) = j(x')$, then $x' - x \in \mathrm{Ker}(j) = \mathrm{Im}(i) \implies x' - x = i(a') \iff x' = x + i(a')$. Ultimately, a' will also be quotiented to the same element in the homology group, because $i(a + \partial a') = \partial(x')$.
- Were we to choose $c + \partial c' \in [c], j(x') = c'$, then $c + \partial c' = j(x + \partial x')$. But since $\partial(x + \partial x') = \partial x, a$ will ultimately be left unchanged.

Checking that this is a homeomorphism is routine. We now finally prove that the sequence of homology groups produced is exact:

- $\mathrm{Im}(i_*) \subseteq \mathrm{Ker}(j_*) : ji = 0 \implies j_* i_* = 0$, and we are done.
- $\mathrm{Ker}(j_*) \subseteq \mathrm{Im}(i_*)$: Let $[b] \in \mathrm{Ker}(j_*) \implies j_*[b] = 0 \implies j(b) = \partial_n c', c' \in C_n(X,A).c' = j(b'), b' \in C_n(X)$. We pick up $a \in C_n(A)$ such that $i(a) = b - \partial b'$ after verifying that the right-hand side vanishes after being acted on by j. Now, $i_*[a] = [b - \partial b'] = [b]$.
- $\mathrm{Im}(\partial) \subseteq \mathrm{Ker}(i_*) : i_*\partial([c]) = [\partial b] = 0 \implies i_*\partial = 0$, and we are done.
- $\mathrm{Ker}(i_*) \subseteq \mathrm{Im}(\partial) :$ Let $[a] \in \mathrm{Ker}(i_*) \implies i_*([a]) = 0$. Then, $i(a) = \partial b$ for some $b \in C_n(X)$, so that $\partial[j(b)] = [a]$.(This will make sense because $\partial j(b) = j(\partial b) = ji(a) = 0 \implies j(b) \in \mathrm{Ker}(\partial_n)$.)
- $\mathrm{Im}(j_*) \subseteq \mathrm{Ker}(\partial) : \partial j_*([b]) = \partial[j(b)] = 0$, since b is a cycle $\implies \partial b = 0$, and we are done.
- $\mathrm{Ker}(\partial) \subseteq \mathrm{Im}(j_*) :$ Let $c \in \mathrm{Ker}(\partial) \implies \partial[c] = [a] = 0 \implies a = \partial_n(a'), a' \in C_n(A)$. Then, one can easily verify that $[c] = j_*([b - i(a')])$ (one can check that the representative is a cycle).

□

Finally, the following result will prove useful.

Lemma 1.9. *If two maps $f, g : (X, A) \to (Y, B)$ are homotopic through maps of pairs $(X, A) \to (Y, B)$, then $f_* = g_*$ on the relative homology groups.*

Excision

Definition. *For a space X, let $\mathcal{U} = \{U_j\}$ be a collection of subspaces whose interiors form an open cover of X. Then, the subgroups $C_n^{\mathcal{U}}(X)$ of $C_n(X)$ consisting of chains $\sum n_i \sigma_i$ such that each σ_i has its image contained in some set in the cover form a chain complex. We denote its homology groups by $H_n^{\mathcal{U}}(X)$.*

Lemma 1.10. *The inclusion $i : C_n^{\mathcal{U}}(X) \to C_n(X)$ is a chain homotopy equivalence.*

Proof. What we want to do here is construct a chain map $\rho : C_n(x) \to C_n^{\mathcal{U}}(X)$ such that $\rho \circ i, i \circ \rho$ are both chain homotopic to identity. Then, by things which have been established previously, it will follow that both $\rho \circ i, i \circ \rho$ are identity at the level of homology groups, from which it follows that i_* is an isomorphism with inverse ρ_*.
Here is a natural candidate for ρ : Given a $\sigma \in C_n(X)$, i.e., $\sigma : \Delta^n \to X$, perform barycentric subdivision upon the n-simplex (say, $m(\sigma)$ times) until each subdivided n-simplex lands entirely in some U_j. This will be a finite process; simply define $\rho(\sigma)$ as the finite sum of these subdivisions. The problem with defining ρ in this manner is that it will *not* be a chain map (this can be checked). □

Theorem 1.11. *Given subspaces $Z \subseteq A \subseteq X$ such that the closure of Z is contained in the interior of A, the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_n(X - Z, A - Z) \to H_n(X, A)$ for all n.*

Proof. Let $\{A, B\}$ be a cover of X. We show that $H_n(B, A \cap B) \cong H_n(X, A)$.
Consider the inclusion map $i : C_n(A + B) \to C_n(X)$, where $C_n(A + B) = C_n^{\mathcal{U}}(X)$, its chain-homotopic inverse ρ and the chain homotopy D. As per their construction in the above lemma, they all take chains in A to chains in A. Thus, we have a quotient inclusion map $i_q : C_n(A + B)/C_n(A) \to C_n(X)/C_n(A)$, as well as quotient maps D_q, ρ_q. They will continue to satisfy the relevant homotopy relationship, so that i_q continues to induce an isomorphism on homology.
Lastly, note that the map $C_n(B)/C_n(A \cap B) \to C_n(A + B)/C_n(A)$ induced by inclusion is an isomorphism at the level of groups itself, since both groups are free with basis singular n-simplices in B and not in A.
Composing the two maps will give us the required homology group isomorphism. □

Corollary 1.11.1. *Let X be a topological space and A be a nonempty closed subspace that is a deformation retract of some neighbourhood in X. Then, the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $q_* : H_n(X, A) \to H_n(X/A, A/A) \cong H_n(X/A)$.*

Proof. Let V be the neighbourhood in X that deformation retracts to A. Using lemma 1.9 and excision judiciously, it is easy to check that the following commutative diagram has isomorphisms for all the horizontal maps.

$$
\begin{tikzcd}
H_n(X,A) \arrow[r] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X-A,V-A) \arrow[l] \arrow[d, "q_*"] \\
H_n(X/A,A/A) \arrow[r] & H_n(X/A,V/A) & H_n(X/A-A/A,V/A-A/A) \arrow[l]
\end{tikzcd}
$$

The right-hand vertical map is an isomorphism because it restricts to a homeomorphism on A^c. Commutativity implies that the left-hand one is an isomorphism. □

Remark. We thus have the following long exact sequence:

$$... \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to ... \to H_0(X/A) \to 0$$

Discussion. Whenever we have a map $f : (X,A) \to (Y,B)$, the following diagrams are commutative:

1.
$$
\begin{tikzcd}
... \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & ... \\
... \arrow[r] & H_n(B) \arrow[r, "j_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & ...
\end{tikzcd}
$$

2.
$$
\begin{tikzcd}
... \arrow[r] & H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "j_*"] \arrow[d, "f_*"] & H_n(X/A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & ... \\
... \arrow[r] & H_n(B) \arrow[r, "j_*"] & H_n(Y) \arrow[r, "j_*"] & H_n(Y/B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & ...
\end{tikzcd}
$$

In other words:

1. The long exact sequence in theorem 1.8 is **natural.**
2. The long exact sequence in corollary 1.11.1 is **natural.**

Degree

Definition. *The degree of a continuous map $f : S^n \to S^n$, denoted by* $\deg(f)$*, is the integer d such that for $f_* : H_n(S^n) \to H_n(S^n), \alpha \mapsto d\alpha$.*

Discussion. Recall that the homology group of S^n is trivial everywhere except at n, where it is $\mathbb{Z}$. f_* is, being a homomorphism from $\mathbb{Z}$ to $\mathbb{Z}$, forced to be of the above form. Some basic properties of degree are as follows:

- deg(1)=1
- deg (f)=0 if f is not surjective (because $f_* = 0$; this will follow from the fact that S^n minus a point is contractible).
- $f \cong g \implies \deg(f) = \deg(g)$
- $\deg(fg)$=$\deg(f)\deg(g)$
- f is a homotopy equivalence $\implies \deg(f) = \pm 1$
- $\deg(f) = -1$ if f is a reflection.
- $\deg(f) = (-1)^{n+1}$ if f has no fixed points.

Discussion. Let $f : S^n \to S^n$ be such that $f^{-1}(y) = \{x_1, ..., x_m\}$, and each point has neighbourhood U_i, with $f(U_i) \subseteq V \ni y$.
The commutative diagram below will allow us to define the *local degree* of f at x_i, denoted $\deg f|x_i$.

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\approx"'] \arrow[d, "k_i"] & H_n(V, V - y) \arrow[d, "\approx"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"'] \arrow[ul, "\approx"] & H_n(S^n) \arrow[u, "\approx"']
\end{tikzcd}$$

p_i, k_i are induced by inclusion.
The two isomorphisms in the upper half of the diagram are due to excision: For example, in the first one, we set $A = S^n - x_i, Z = S^n - U_i$, and check that $\bar{Z} \subset \mathrm{Int}(S^n - x_1)$.
The two isomorphisms in the lower half are due to lemma 1.9.
With this, we have $\mathbb{Z} \cong H_n(U_i, U_i - x_i) \cong H_n(V, V - y)$, and the local degree will be said to be the degree of f_*.

Theorem 1.12. $\deg f = \sum_i \deg f|x_i$.

Proof. From commutativity of the above diagram and the decomposition $H_n(S^n, S^n - f^{-1}(y)) = \bigoplus_i H_n(U_i, U_i - x_i)$: $p_i(j(1)) = 1 \forall i \implies (1, 1, ...1) = j(1) = \sum_i k_i(1), f_*(k_i(1)) = \deg f|x_i \implies \deg f_* = f_*(1) = f_*(j(1)) = \sum_i \deg f|x_i$. □

Example. $f : S^1 \to S^1, z \mapsto z^k$ is a map of degree k. (Elaboration)

Theorem 1.13. $\deg Sf = \deg f$*, where* $Sf : S^{n+1} \to S^{n+1}$ *is the suspension of* $f : S^n \to S^n$*.*

Proof. First, the following commutative diagram:

$$\begin{tikzcd}
CS^n \arrow[r, "Cf"] \arrow[d, "q"] & CS^n \arrow[d, "q"] \\
S^{n+1} \arrow[r, "Sf"] & S^{n+1}
\end{tikzcd}$$

We have used the fact that $SS^n \cong S^{n+1}$. Cf, Sf are the obvious maps.
And now, this:

$$\begin{tikzcd}
H_n(CS^n) = 0 \arrow[r] \arrow[d] & H_n(S^{n+1}) \arrow[r, "\partial \cong"] \arrow[d, "Sf_*"] & H_n(S^n) \arrow[r] \arrow[d, "f_*"] & H_{n-1}(CS^n) = 0 \arrow[d] \\
H_n(CS^n) = 0 \arrow[r] & H_n(S^{n+1}) \arrow[r, "\partial \cong"] & H_n(S_n) \arrow[r] & H_{n-1}(CS^n) = 0
\end{tikzcd}$$

We have used corollary 1.11.1 and the fact that $CS^n/S^n \cong S^{n+1}$. That the boundary map is an isomorphism follows from exactness. Since naturality implies that this diagram also commutes, we have the desired result. □

Mayer-Vietoris sequences

Discussion. Let $\{A, B\}$ be open sets in X whose interiors form a cover. Then, the following is a short exact sequence:

$$0 \to C_n(A \cap B) \xrightarrow{\phi} C_n(A) \bigoplus C_n(B) \xrightarrow{\psi} C_n(A + B) \to 0$$

where $\phi(x) = (x, -x)$ and $\psi(x, y) = x + y$.
Now, following a procedure identical to the one in theorem 1.8 and using the fact that $i_* : H_n(A+B) \to H_n(X)$ is an isomorphism (lemma 1.10), we have the following long exact sequence:

$$... \to H_n(A \cap B) \xrightarrow{\phi} H_n(A) \bigoplus H_n(B) \xrightarrow{\psi} H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to ...$$

This long exact sequence is known as a Mayer-Vietoris sequence.

Cohomology Groups

Discussion. Let X be a topological space, and $C_n(X)$ be the free abelian group with basis being the set of singular n-simplices.
Define $C^n(X) := \text{Hom}(C_n(X), \mathbb{Z})$.
Define the **coboundary map** as $\delta : C^n(X) \to C^{n+1}(X), \psi \mapsto \psi\partial$, where $\partial : C_{n+1}(X) \to C_n(X)$ is the relevant boundary map. (Note that $\partial^2 = 0 \implies \delta^2 = 0$.)
We now have the following 'dualized' cochain complex:

$$... \xleftarrow{\delta_{n+1}} C^{n+1}(X) \xleftarrow{\delta_n} C^n(X) \leftarrow ...$$

The nth **cohomology group** $H^n(X)$ is then defined as $\text{Ker}(\delta_{n+1})/\text{Im}(\delta_n)$.
Elements of $\text{Ker}(\delta)$ are called **cocycles**, and elements of $\text{Im}(\delta)$ are called **coboundaries**.