Chapter 1

Some metaphysical (meta-theoretic?) commitments: State of affairs, judgement (scheme), obtainment (?). Some more, I think theoretic ones: Object, property (reducible to relation?), relation, category (of object) (example: point, line, plane— "immediately given"/exhibited in intuition). A hypothesis: The characteristic features of a category of objects, which can be known via existential judgements, determines its extension.

Weyl sets up his deductive system.

Primitive judgements (immediately given) and complex judgements (constructed). Pertinent, general and particular judgements. If only one object satisfies a judgement scheme with one blank constructed without any "filling in", we call it an *individual*. (Including Pr. 5 leads to applicability of the notion of 'individual' to object, making it vacuous.)

Natural numbers are a domain of individuals. "It is impossible for a number to be given otherwise than through its position in the number sequence." Thus, they are characterized by means of that property (via succession). It is shown then that 1+2=3 is a general judgement. "In some cases, for example in arithmetic, all objects of the category under consideration are 'individuals'."

On the other hand, we can have the judgement scheme true of either *each* or *no* object in the concerned category; we then call our category *homogeneous* (an example: a point in Euclidean geometry).

Weyl continues setting up the syntactic system of logic used: Self-evident judgements, absurdities, implications, equivalence are passed over. These properties are concerned with the syntactic structure of the judgement, and not the category or object it deals with (nor the extension of the relation). N.B.: Judgement is a meta-theoretic entity. So, his views on the proper use of the axiomatic method are subjectivistic, and he will probably refer you to Husserl for more.

"Mathematics concerns itself with *general, pertinent, true* judgements." A formal mathematical proof concentrates on the logical inferences involved from a set of axioms, and not the nature of the object or the state of affairs being affirmed.

Weyl's is not a set theory, but a theory of relations-in-extension. Do we move now to the 'model' (or semantics)?

Properties and relations and their definition principles are now 'imported' to form sets: "The mathematical process". The properties/relations are defined *intensionally,* and the sets are defined *extensionally*. Weyl would propose infinitely many extensionality axioms for identity here. "The transition from the 'property' to the 'set' signifies merely that one brings to bear the *objective* rather than the purely logical point of view, i.e., one regards the *objective* correspondence (that is, 'relation in extension', as logicians say) established entirely on the basis of acquaintance with the relevant objects as decisive rather than logical equivalence."

The abstraction process is distinct and independent from the mathematical process (the latter being the identification criterion).

Dirichlet defines a function by the property that "to any x there corresponds a single finite y". Weyl, the constructivist, thinks this is entirely too vague; one must indicate a rule, a characteristic, and not an arbitrary infinite gathering—for "inexhaustibility" is of the essence of the infinite. The central problem of the book, unsolved since Pythagoras, is now stated: The continuity of the intuitive continuum and the atomism of the mathematical continuum. We have to now bridge the gap between the intuitive finitism and the infinite. (Cantor and Dedekind presupposed the infinite and developed their mathematics, missing the fact that the infinite must first be constructed.)

We cannot form power sets unless we can form the corresponding sentence matrix first.

Weyl's minimal basic category of objects is the natural numbers, and only primitive relation is succession: This may be called 'arithmetism'. And so pure number theory may be said to form the centerpiece of mathematics. We add for them a specifically mathematical principle of definition, the principle of iteration, whose general definition will be given only later but which is essentially to give rise to multiplication.

Richard's antinomy: The possible combinations of finitely many letters form a denumerable set. Since each definite real number can be defined by finitely many words, there are only denumerably many reals.

All possible judgement schemes (properties and relations) derived from our "production process" (including the simple ones) can be enumerated in a sequence. Since each one corresponds to a set of natural numbers, all possible sets of natural numbers must also be denumerable in a sequence.

This is in logic. On the other hand, when we move to the domain of *mathematics* (the extensional), we can apply Cantor's proof—which, in turn, fails in our antecedent-to-mathematics sphere of operation because we cannot *construct* a judgement scheme which is a binary numerical relation R(xy) such that for every set of numbers (corresponding to some property) there is a number *a* such that that the one-dimensional set associated with R(xa) is identical to the original set (R(xa)=P(x)). Cantor's proof hinged on the set of numbers which do not stand in R with themselves (~R(xx))—for these will not be a part of any set (associated with any P(x)).

"If we adopt the concept of denumerability suggested by this proof, then *naturally there is no reason at all to assume that every infinite set must contain a denumerable subset*—a consequence from which I certainly do not shrink." The point is that the infinitude of a set does not guarantee the existence of a *constructive* relation (derivable from our finite definitions in #2) between a subset of it and N.

"The category 'natural number' together with the primitive relation S associated with it we call the *absolute* sphere of operation."

Sets of the first level: Generated by the mathematical process from the primitive and relations. Category of the set is determined by number of blanks & category of objects afforded of by each blank.

Sets of the second level: The relation 'ϵ' is introduced. On the RHS blank it affords of a certain category of first-level sets, while on the LHS blanks it affords of those categories of objects afforded of by the set on the RHS. The set generated by this relation is of the second level.

In a second-level relation, the existence concept seems to apply equally to first-level sets and objects of the basic categories. We restrict its application to only blanks affiliated with a basic category. This is so that the second-level relation cannot introduce any new relations between our basic objects, the possibility of which threatens paradox (the example which I think suits: the set of objects not in any set'); "the objects of the basic categories remain uninterruptedly the genuine objects of our investigation only when we comply with the narrower procedure; otherwise, the profusion of derived properties and relations becomes just as much an object of our thought as the realm of those primitive objects".

Note: This is not a general restriction on the formation rules of his *language*. If our sentence matrix has a first-level object as a free variable in a blank, then we may of course import it and say without violence that there exists a second-level set corresponding to it. It is just that the sentence matrix itself cannot bind *any* higher-level variable to begin with.

The existence of a higher-level set is linked with the existence of a lower-level set satisfying a certain property, and so on unto the lowest level.

We may easily construct the rational numbers and two further relations to add to our underlying sphere of operation. Finally, following Dedekind, a real number is defined as a (one-dimensional) set of rational numbers (those smaller than it). Weyl shows this explicitly.

Do we go ahead and construct a "hierarchical" version of analysis (for there are connections *between* real numbers too: As illustrated, the supremum of a set of real numbers becomes a second-level real number). We usually suppress the level and tear through analysis as if they were all equivalent, but this is may just give rise to circularity. What now?

"The only natural strategy is *to abide by the narrow iteration procedure.*" The proposition asserting the existence of a supremum must be abandoned.

Note: The ramified theory of types is weaker and restricts binding to within a certain level, but this breaks up the continuum—the supremum of a set becomes an object at a different 'level' than another real number in the set; indeed, since the set may contain the supremum, we would have two extensionally identical species occupying distinct levels (and made unnecessary intensional distinctions).

What is a function?

Option A: A binary relation whose one blank *x* is affiliated with real numbers (one-dimensional sets of rational numbers) and other blank with some category k (e.g., the natural numbers) such that for every t, there is only one x (t is the independent variable).

But upon this formulation, we cannot even define the sum of two functions without using 'there is' to fill a blank which is not affiliated with the basic categories.

Option B ("seems more natural"): *x* is a set of rational numbers characterized by some property they have in common. If object *t* in category *k* is a "component of this property", then *x* depends on/is a function of *t*. *t* is a "component of this property" if the property arises by filling in the second blank of a binary relation whose first blank is affiliated with the category of rational numbers with *t*. The set of rational numbers corresponding to this property S (__, *t*) is a function of *t*. This property 'prefigures' the notion of a function—it generates a set of rational numbers, and may be a real number for every *t* (in which case it is recognizable as the everyday notion of a function). And in this case, the sum of two functions can be said to be a function in a well-founded manner (S requires us to apply the existence concept only to the basic categories).

The generalized version of this is presented: "To every delimited relation whose blanks are divided into two ordered groups, there corresponds a function. If each independent blank is filled by some objects of the appropriate category, then the set which corresponds to the relation that thereby arises is the value of the function for the system of arguments used in filling the blanks." For Weyl, the value of a function *can't* be a basic object, and is *always* some set (but of course, a *unique* set, thus ensuring that for every argument a function has only one value).

Our system is significantly strengthened; we can characterize complementation, among other things. The sets become borderline cases of functions: those where the number of independent variables has been reduced to 0.

The principle of substitution: A second-level relation may be substituted by its first-level explication. (Not the other way round; that would defeat the purpose.)

The principle of iteration: If the dependent and independent variable are of the same category, we can put the dependent variable back into the independent variable to get an iterated function *n* number of times, where *n* is a natural number.

Three extensions/generalizations to the principle of iteration: We can have blanks in R unaffected by iteration; we can effect iteration upon multiple blanks simultaneously; the function substituted in the *n*th iteration can itself depend on *n*.

It is shown how the cardinality function may be constructed using the principle. Finally, Weyl remarks that the principle of iteration carries with it/is accompanied by the Bernoullian "inference from n to n+1" (inference by complete induction) (a logical form of inference).

Defining natural numbers on the basis of sets is 'unnecessary and deceptive', for it is the former which are known through "pure" intuition.

Chapter 2

From our basic sphere of operations (the natural numbers and the successor function), we define addition and multiplication. Note that Weyl includes an induction axiom in order to prove certain basic properties of addition and multiplication.

"We call a (one-dimensional) set of numbers a *cut* of the number sequence if there are no two numbers *m* and *n* such that *m*<*n* and *n* is, but *m* is not, an element of the set. *If A is a cut which is neither the empty set nor the universal set, then there is a number n such that A coincides with the set of all numbers <n+1.*" The italicized statement is proven shortly. We call *n* the cardinal number of the cut.

Some theorems about cardinality:

1. If a set consists of at least *n+1* elements, then it also consists of at least *n* elements.
2. If a set does not consist of at least *m* elements, then neither does it consist of at least *m+n* elements.
3. The cardinality of a part is no larger than the cardinality of the whole.
4. If an arbitrary element is removed from a set of cardinality at least *n+1,* then the cardinality of the new set is at least *n*.
5. The cardinality of the union of two disjoint sets is equal to the sum of the cardinality of each set.

These are all proven using induction (the fourth one requires an additional lemma stating that substituting a new element leaves the cardinality of a set unchanged).

"We can establish that the cardinal number *p(n)* of the prime natural numbers up to *n* is a *function* of *n* in our precise sense."

Q^+ is constructed as a two-dimensional set of natural numbers and some basic properties of fractions are exhibited. Weyl now says that we can admit both natural numbers and fractions as basic categories from the start without any violence. Finally, Q is constructed as a four-dimensional set of natural numbers.

A real number is characterized as an open cut of rational numbers. Addition and some other basic things are defined on this basis. Our logical principles have led up to algebraic ones. The concept of an algebraic number is defined constructively.

The real numbers are fundamentally different from the ones preceding it, because *one* real number is a four-dimensional set of natural numbers. (On the other hand, a *domain* of rational numbers is a four-dimensional set of natural numbers.)

The *limit inferior* of a sequence of functions (from N to R) is defined: It is the domain of rational numbers—or rather, an open cut of them—which stops at that rational beyond which there is no rational number such that the relation between n and r on which the function is based holds between it and *all* m>n (for some n). (I think you'll have the constructive definition of lim sup if you exchange the quantifiers on m and n.)

Convergence is defined via Cauchy's convergence principle, and the number to which it converges is identified as the lim inf and called just the limit. The completeness of R is stated in various equivalent ways; it is pointed out that not all are legitimate in constructive analysis.

We go over the Heine-Borel theorem. We construct series as a sequence of partial sums.

The domain of a function is a delimited set, but the range is not. (The latter is obtainable only by examining the independent variable, which is a real number.) Continuity (as well as uniform continuity) is defined in pretty much the orthodox way (but epsilon, delta are restricted to be fractions, for some reason I suppose will be clear soon). They are not delimited properties.

The intermediate value theorem and some associated statements are proven (constructively).

Since continuity is a delimited property, it may fail to whole if we count as reals other kinds of sets of rational numbers (and not just open cuts).

But such an expansion of definition is absurd when it comes to the intuitive, temporal continuum. So we must try to justify why the particular definition we have allowed for real numbers creates a continuum which matches up with the intuitive continuum.

Weyl grants three assertion to a mathematical theory of time (the intuitive continuum): A basic category (time-point), a binary relation (earlier-than/later-than), and a quaternary relation (that of two time spans—each one defined by two time points—being equal).

The two continuums would match up if the following conditions are satisfied:

1. "During a certain period" is equivalent to "In every time-point which falls within a certain time-span".
2. A time-point is a real number (in the usual sense of the latter) and vice-versa.

The "intuition of time" leaves us only more confused, and certainly provides no answer as to whether the two hold or not. But it does give us two facts:

1. An individual point in the temporal continuum is "pure nothingness when taken by itself".
2. It is of the essence of time that a fixed time-point can only be approximated.

This holds for any intuitively given continuum (for example, space).

"The intuitive and mathematical continuum do not coincide; a deep chasm is fixed between them. So one might say that our construction of analysis contains a theory of the continuum which must establish its own reasonableness in the same way as a physical theory."

In order to be sufficient, such a theory, then, must satisfy the following:

1. An isolated point cannot be individually characterized. This is circumvented by the introduction of the coordinate system: "A final scanty token in this objective sphere that existence can only be given as the intentional content of the processes of consciousness of a pure, sense-giving ego."
2. There must be a bijection between R and T. This 'continuity axiom'/'transfer principle' is what will give judgements a clear sense in spite of 1.
3. Suppose we create a new foundation called 'hyperanalysis', which admits reals as a new basic category. This differs from what we have done; more sets of real numbers are there in it. In general, the various theorems that we have proven would not hold as stated (for us, they are only satisfied by the functions and sequences we could construct in our way). But if we go ahead with this new theory anyway, the question arises as to how we can rewrite #2 (which is necessary to develop anything beyond elementary geometry): For the bijection would hold not only between points and real numbers, but also between sets of the two, and sets of sets of…and so on.

Let us analyze the aforementioned 'transfer principle' in more detail.

Construct a relation L using the principles of definition such that for a time span OE, the time-point P uniquely satisfies L(OEP), and if L(O'E'P'), then P' stands in the same *ratio* to O'E' as P does to OE. Furthermore, if there exists another relation L' such that L'(OEP), then L=L'. Now, the extensions of the relation L (which we are calling "ratios") can be one-one represented by real numbers.

Having conceptually established the elements of time, we now construct a magnitude/measure.

Our category of objects is time spans, and our primitive relations are equality and addition (with inequality being a derived relation). Our field of operations is *homogeneous*: A judgement scheme is either true of each time span or of no time span. Thus, spans can be determined only relative to one another. R(a,b)⇔R(c,d) if a=c, b=d. If two relations have equal extension, we replace them with the 2D set of spans corresponding to it. We express this by the formula b=aL; b stands to a in the ratio L; L is the measure. We set equality to be the unit ratio.

We can show that two spans stand in one and only one ratio to one another; and that the "measures" coincide with the positive real numbers.

We move from time to geometry.

A line is distinguished from a curve. "It is essential to a curve that it be exhibited only in a movement. The continuum of path-points *spreads over* the continuum of time points in a continuous monotone manner. This conception allows the path to be separated from the movement which produces it."

The concept of a surface is analogous to that of a curve. How does a parametric representation, which, only by tearing the continuum apart into isolated points, gives us the surface-points in terms of the space-points it is constituted of, grasp their continuous connectedness?

Weyl basically refers to the notion of a closure point for this, and says that a "surface in itself" is given if every space-point is a closure point (has other surface points "infinitely close" to itself). (Squares of smaller and smaller size are used here, instead of circles/balls.)

A flaw in this attempt to link the genuine continuum with the real continuum lies in the fact that we can choose our neighborhood in the latter to be of any arbitrary shape.

Finally, the conditions under which two analytic spatial surfaces coincide (i.e., represent the same spatial surface in the intuitive sense) is given.

Appendix

The vicious circle of analysis:

The *sense* of a *concept* defines a *sphere of existence* which the *objects* sharing the *essence* expressed in the concept are *assigned*. Loosely, Concept->Essence->Sense->Sphere of existence->Object. But the concept may or may not be *extensionally determinate* ("intrinsically clear"). "The sense of a concept is logically prior to its extension." Only if it is can we pose existential questions. And it is for the concept "natural number". However, "property of natural numbers" is not. And so, "The concept 'real number' is not extensionally determinate."

"It is extraordinarily unlikely that it is possible, in an exact way, to set down an extensionally determinate concept '**k**-property' such that each property P, whose definition involves the *totality* of **k**-properties as indicated above, is extensionally equivalent to a **k**-property. In any case, *not even the shadow of a proof* of such a possibility exists; but precisely this proof would have to be effected in order for the assertion of the least upper bound's existence to *receive a sense in all cases and be universally true*."