[i]Symmetrie

by Hermann Weyl

Chapter 1

"We begin with some general but vague principle, then find an important case here we can give that notion a concrete precise meaning, and from that case we gradually rise again to generality, guided more by mathematical construction and abstraction than by the mirages of philosophy' and if we are lucky we end up with an idea no less universal than the one from which we started. Gone may be much of its emotional appeal, but it has the same or even greater unifying power in the realm of thought and is exact instead of vague."

Symmetry has turned out to be an important idea. In origin, it looks to be something man abstracted intuitively as a certain common feature between objects in nature (not necessarily material) which simultaneously manifest the duality of order and beauty. Subsequently, it appears that, in a reverse-engineering of sorts, the ancients declared that the celestial bodies must be perfectly symmetric geometric figures, for anything else would detract from their supposed order and beauty.

In everyday language, this abstraction is associated most closely with the notion of bilateral symmetry. A body is bilaterally symmetric with respect to a plane if reflection across the plane carries it into itself.

An interesting face illustrated is that historically, this type of symmetry is associated in art and life with the idea of order and law; the presence of asymmetry in many paintings is often a deliberate signifier of arbitrariness and freedom.

"To the scientific mind, there is no polarity between left and right: Reflection in a plane is an automorphism." (More on automorphisms later.)

Newton believed in absolute spacetime, and argued that it implied the existence of God, for only He could have decided that matter move in the direction it does, rather than its opposite. Leibniz refuted this as saying it violated the principle of sufficient reason, and concluded that there must be no intrinsic difference between left and right.

"Scientific thinking sides with Leibniz. Mythical thinking has always taken the contrary view as is evinced by its usage of right and left as symbols for such polar opposites as good and evil."

This problem of equivalence arises not only with direction but with time and electricity as well, wherein it is clearer that empirical evidence is needed to settle it.

Now, if our physical laws are symmetric, we can understand why nature is symmetric: a state of equilibrium is likely to be symmetric; "under conditions which determine a unique state of equilibrium the symmetry of the conditions must carry over to the state of equilibrium". I see this as a loaded comment: We abstracted symmetric laws from nature and apply it to predict a symmetric nature; from the phenomenal to the abstract and back to the phenomenal.

Phylogenetic evolution induced asymmetries in organisms. The existence of asymmetry "proves that *contingency* is an essential feature of the world" (in the sense of contingency when we make the trip back from the abstract to the phenomenal; the deeper mode of contingency—the one occurring in the first act of abstracting from the phenomenal—is never thought of.)

And it is in this contingency that the principle of sufficient reason seems to fail.

Organisms often preferentially select one laevo/dextro form over the converse, leaving an asymmetry in the abundances of the two in the world. Although he is probably wrong, Pasteur felt this asymmetry to be of such importance that he believed that life alone could induce this asymmetry, and that life was defined as the ability to induce this asymmetry.

Weyl declares that, in any case, this asymmetry cannot be a foundational principle of biology but, rather, must be due to some contingent occurrence causing deviation from symmetry.

Now, moving on from bilateral (a)symmetry in phylogenesis to the same in ontogenesis: Do we have left-right polarity in the process? Moreover, what determines the plane across which this polarity exists?

This plane—the plane of first division—is determined by the polar axis of the egg and the entry direction of the sperm. The primary polarity (the polar axis of the egg) as well as the subsequent bilateral symmetry (induced by the determination of the plane when the sperm enters the egg) are often determined by external factors.

Based on a few experiments, Weyl offers the following skeleton of an answer for the first question: That an equivalent potentiality exists for all functions in both the left and the right, but that certain factors (usually genotypic, sometimes phenotypic) cause only one to manifest; and under abnormal conditions, the asymmetry may be reversed.

Chapter 2

Weyl defines automorphisms "as those transformations which leave the structure of space unchanged." It is worthwhile to note that as such, endomorphisms are structure-preserving maps with equal domain and co-domain; automorphisms are a special class of those which also have an inverse. Possibly, the idea of endomorphisms did not exist in frequent usage when Weyl wrote the book; I see that the first usage of the term is 29 years before the first publication of Symmetrie. Alternatively, the idea of a map without an inverse is considered senseless in Euclidean space.

In any case, automorphisms are that which were called similarity transformations of space by the geometers.

Any given automorphism forms an equivalence class over space. The set of automorphisms satisfies the group axiom, and forms a group.

Now let alone endomorphisms—Newton and Helmholtz were even stricter than automorphisms and associated the structure of space with the subgroup of automorphisms without change of scale: Congruences.

Congruences without reflections may be called proper congruences; and the ones with, improper congruences. Note that only the former forms a group (two reflections is a proper congruence). Rotations around a given center O form a group, and the elements of this group may be proper or improper congruences. Translations are the simplest type of congruences.

The symmetry group of a spatial configuration is given by the set of automorphisms which leave it unchanged, i.e. whose action on it is the same as that of identity.

Finite figures can only have congruent automorphisms in their symmetry group, unless the figure is a point. (Otherwise, we could iterate the automorphism to make the figure arbitrarily large, contradicting the fact that it is finite and thus bounded.) By the same argument, any finite subgroup of automorphisms can contain only congruences, since if there were any scaling map, the group would contain all its infinite iterated versions.

Moral of the story: Congruences are important.

Reflection in any of its points constitute the only improper congruences of the 1D line, while translation are its only proper congruences. "A figure which is invariant under a translation shows what is in the art of ornament called "infinite rapport", i.e. repetition in a regular spatial rhythm." The figure would also be invariant under any iteration of the *t* as well as its inverse.

The two possible symmetries for a 1D pattern are the rhythmic and the reflexive (the former relating to translation and the latter to reflection).

In 2D, we often see in nature the combination of translation + *longitudinal* reflection symmetry, although the former (bilateral symmetry) is much more common. Weyl then muses on the fact that in music, temporal repetition (rhythm) is much more important than reflection (inversion in time).

We now move from translatory symmetry in one dimension to rotational symmetry in three, obtained by slinging the object around an imaginary cylinder (or any object with cylindrical symmetry, i.e., carried into itself by all rotations about a certain axis) with appropriate circumference.

Any translation in a line figure with translatory symmetry is an iteration of the fundamental one; any rotation in a figure with rotational symmetry is an iteration of the fundamental one.

Some examples of cyclic symmetry in nature are given. Pentagonal symmetry is prevalent in the organic world, although it is not present among inorganic crystals.

Now, we move from proper to improper rotations (rotoreflections—recall that an improper congruence was one which interchanged left and right). "C1 means no symmetry at all, D1 bilateral symmetry and nothing else." Weyl points out that most of the proper rotations we have considered have also happened to have the rotoreflection symmetry, and offers examples of objects with exclusively rotational symmetry. (It is also possible, of course, for an object to have rotoreflectional but not rotational alone.)

Weyl now finally considers the dilatation, followed by rotodilatations (the logarithmic spiral), followed by rototranslations (the helix).

Weyl goes on a brief excursion about continuous groups and their discontinuous subgroups, showing how a continuous transformation function (describing a "fluid") can be broken up into discontinuous segments *if* it describes uniform motion. In fact, it looks to me that this is the first real invocation of the notion of temporality.

The third, says Weyl, is the most general rigid motion in 3D. The fraction that the angle of rotation of (in the helix) is of 2pi has often followed the Fibonacci sequence, the expansion of the number that is the golden ratio. "This number…has played such a role in attempts to reduce the beauty of proportion to a mathematical formula."

To sum up: We have considered translation, reflection, rotation and dilatation (and some combinations of theirs).

We have seen that, apart from reflection, all the symmetry groups considered so far are described by iterations of one operation. Furthermore, in these groups, the rotational symmetry group alone is finite (excluding the infinitesimal angle, of course).

In 2D space, we have a figure corresponding to each order of rotational symmetry (triangle, square, pentagon…). This is not the case in 3D space, where we have only five regular polyhedral: The five Platonic solids. (Tetrahedron, cube, octahedron, dodecahedron, and icosahedron.) "The discovery of the last two is certainly one of the most beautiful and singular discoveries made in the whole history of mathematics." An interesting footnote is that Plato saw the one of the four elements in all but the dodecahedron, in which "he sees in some sense the image of the universe as a whole". Even more interesting a footnote: "A. Speiser has advocated the view that the construction of the five regular solids is the chief goal of the deductive system of geometry as erected by the Greeks and canonized in Euclid's *Elements*." Kepler also had some funny ideas about reducing the distances in the solar system to alternatingly inscribed and circumscribed Platonic solids.

Now, let us see the proper rotation groups corresponding to these solids.

But first, we reformulate the 2D proper rotations in three dimensions.

With the introduction of a third dimension, rotoreflections can also be converted into proper rotation groups about the vertical and horizontal axes. In general, each D'n is a group consisting of rotation around the vertical axis by 360/n and 180 rotation about horizontal axes (reflection), which are inclined at 360/2n to each other.

Evidently, the modified D'1 is identical to C2, since they both consist of only 180 rotation about a line. We thus eliminate the former. Furthermore, the D'2 is a four-group consisting of identity and reflection about the 3 perpendicular axes.

Now, the five Platonic solids give rise to only three new groups. This is because—try and imagine this by circumscribing them—every rotation leaving a cube invariant leaves also the octahedron; similarly related are the dodecahedron and the icosahedron. Thus, we have the groups called W, P, T respectively. With this, we have all the finite groups of proper rotations in 3D.

What of improper rotations in 3D? Let us denote these by Dn.

In 3D, D'n being vertical rotation and horizontal reflection is still proper rotation, whereas Dn being vertical rotation and vertical reflection is improper rotation. (When you have rotoreflection about the *same axis* (an impossibility in 2D) you are left with only one fixed point. My spatial visualization skills are not too good, but the movement from a fixed axis to a fixed point is probably what interchanges left & right and makes it improper.)

For constructing this, we consider the group given by reflection in a point (center of inversion) in 3D. (In 2D, this just amounts to 180 rotation—which is why the improper group Dn in 2D converts to the proper group D'n in 3D.) This operation Z obviously turns left screws to right screws, and commutes with every rotation.

Consider now a finite group T of proper rotations. One way of including improper rotations is by just taking T'=T+ZT. (Imagine: Rotating along the vertical and then inverting amounts to rotating and then reflecting about the vertical axis & plane.)

Alternatively: Take a subgroup T of T' such that half of the elements of T' lie in it. Replace the half which do not lie in T', say S, with ZS. We then get the group T'T which contains T while the other half of the operations are improper. For instance, D'n contains Cn. Replace the non-occurring operations

(180 rotation around the horizontal axes) with their product with Z (which are then reflections in the vertical planes perpendicular to these axes). D’nCn then consists of rotations around the vertical axes and reflections in the vertical planes.

Finally, the list of all the finite groups of rotations is furnished. A more technical handling of the same is offered in the appendices.

The basic idea for the list of all finite groups of proper rotations in 3-space was to converge the following two facts: That each nonidentity operation has two poles, and that each pole has (v-1) nonidentity operations leaving it invariant—where v is the multiplicity of the pole.

Improper rotations: It is first proven that half of the total operations in the total group are proper, while the other subgroup of improper ones form the other half. (This is based on the fact that the composition of two improper operations is proper.) Finally, it is just shown explicitly how we may include a given improper rotation into our group.

Chapter 3

We first analyze ornamental patterns in two dimensions.

The close packing of identical circles forms a figure such that the tangents to each pair of circles forms a hexagon. These hexagons fill up the whole plane without any lacunae, unlike the circles.

A hexagonal filling-up of the plane is one which minimizes contour length for the same area (the 2D equivalent of a sphere being formed due to minimum area for the same volume). Thus, the circular cylinders of the beehive are morphed into hexagonal prisms by the forces of capillarity.

However, it is impossible for a hexagonal net to cover a sphere due to a fundamental formula of topology (for any net of the sphere, the number of units A + the number of lines where two intersect E – the number of points where three intersect C = 2. For a hexagon, E=3A, C=2=>A+C-E=0.)

We move next to close packing (of spheres) in 3D—the cubic closed packing is described. When the spheres are expanded uniformly in order to cover up the lacunae and fill up all of space, they transform into rhombic dodecahedron (not a regular figure, unlike the hexagon). “The bees’ cell consists of the lower half of such a dodecahedron with the six vertical sides so prolonged as to form a hexagonal prism with an open end.”

Darwin says: “Beyond this stage of perfection in architecture natural selection could not lead; for the comb of the hive-bee, as far as we can see, is absolutely perfect in economizing labor and wax.”

Weyl notes that the tetrakaidekahedron is another space-filling figure which gives an even better surface area-volume economy, although whether or not it is the absolute minimum remains unproven.

A 2D “coordinate system” is characterized by an origin and two basis vectors.

A vector space upon which the distance function (which would make it a metric space) has not yet been imposed is an affine space. One can, in essence, compare lengths only of parallel vectors.

Upon this, a positive-definite metric ground form (the distance function) is imposed by Weyl. An *orthogonal transformation* is a transition across coordinate systems which leaves the length invariant (e.g. $x_1^2+x_2^2=x_1'^2+x_2'^2$ for Cartesian coordinate systems).

"But with a slight modification such a transformation may also be interpreted as the algebraic expression of a rotation."

Consider the transformation leaving the vector coordinates (x1, x2) invariant. Then, a vector in the old basis changes into some other vector in the new basis. This new vector in the old basis equals the old vector in the new basis. The transformation is homogeneous linear.

The transformation of an invariant point X is, however, non-homogeneous, and it is congruent if the part of the transformation corresponding to vector-mapping is orthogonal. In this case, we also call the overall non-homogenous transformation orthogonal.

Two transformation matrices are orthogonally equivalent if they differ only by the basis in which they are expressed—in other words, if they are similar. Since we know that similar matrices have the same trace/rank/determinant, we can easily verify the fact that all the operations in the list are orthogonally inequivalent.

With these tools, we describe the list of finite rotation groups in 2D in algebraic terms: It is a complete list of orthogonally inequivalent finite groups of orthogonal transformations.

"The *symmetry of ornaments* is concerned with discontinuous groups of congruent mappings on the plane." This is because the presence of translation makes it impossible for us to impose finiteness, in lieu of which we go for discontinuity instead. We consider the case of two-dimensional infinite rapport (a lattice) induced by discontinuous translations. The linearly independent vectors describing the fundamental translation in each direction are called the lattice basis. Any translation must be a linear combination of these with *integral* coefficients.

The transformation matrix across two lattice basis as well as its inverse must have integral coefficients; alternatively, its determinant must be of modulus unity (these special matrices are called unimodular matrices).

Now, we want to determine all the possible lattice structures in 2D ("all possible discontinuous groups of congruences with double infinite rapport").

Due to the imposition of discontinuity, the rotational groups must be finitely many. The finite group of rotations associated with a lattice is what determines its symmetry class.

Now, the *only* possible rotational symmetries for *any* 2D lattice group, Cn and Dn, have n ranging from 1 to 6 only—and excluding 5. The impossibility of n=8 specifically is shown explicitly.

Any lattice is left invariant by C1 and C2. Consider D1, which is just reflection. This symmetry is possessed by a rectangular lattice as well as a diamond lattice.

After finding the 10 possible rotation groups T and the translation-generated lattices L which they leave invariant, we have to concatenate them to get the full set of all possible 2D lattices. The cardinality of this set, as it so happens, is 17.

What we have to figure out is a way to pin down when two lattices amount to the same case, considering the infinite manifold of possibilities. Thus, we make the move from speaking in terms of an infinite manifold of discrete metric-space lattices to discrete lattices over an infinite metric-space manifold.

Usually, given an affine space, we impose the Cartesian metric ground and subsequently write the algebraic representation of the lattice in terms of variables. We now reverse this chronology; by choosing our basis vectors as the lattice basis, the lattice vectors become just those coordinates which are integers—but the metric ground form, in general, ceases to be normalized.

Consider the case of the rectangular lattice. Setting the lattice basis as our basis vectors, D1 itself contains the identity and the operation which acts as identity upon one vector and flips the other by 180 (quite similar to its usual form, in fact). The metric ground form is now of the form $a_1x_1^2+a_2x_2^2$ with a1 and a2 unequal in general, since in general, the lattice basis vectors will be of unequal lengths (excepting the special case of a square lattice).

For the diamond lattice, however, it looks quite a bit different; indeed, the rewritten D1 consists of interchanging the given vectors. Furthermore, since the sides are equal but the angle is not 90, the basis vectors are of equal length but not orthogonal, and the metric ground form is of the form $a(x_1^2+x_2^2) + 2bx_1x_2$.

D1 splits into two groups of linear transformations (corresponding to each of the two lattices just considered). While these two groups remain orthogonally equivalent, they are no longer *unimodularly equivalent* (two groups are unimodularly equivalent if they can be changed into one another by a unimodular coordinate transformation).

The real question, then, is *how many unimodularly inequivalent finite groups of linear transformations with integral coefficients in two variables exist?*

Just the way D1 broke up into to U.I. L.T., so do D2 and D3. In all, there are *13* of the above. When we now finally introduce translation, we get 17 unimodularly inequivalent forms.

An interesting footnote: "For any finite group of linear transformations with real coefficients one may construct positive quadratic forms left invariant under these transformations." Thus, it is so that for each of the above 13 groups, there exists a continuum of corresponding invariant positive quadratic forms.

The principle of this analysis is highlighted in terms of the discrete-continuous dichotomy: In the metric-adapted ground form, the discrete lattices range over a continuum, whereas in the lattice-adapted ground form, the discrete ground forms range over a continuum.

Another improper congruence in 2D is introduced. Analogous to how Dn stood for rotation followed by reflection, a *gliding axis* stands for reflection followed by translation, packed into one operation.

An example is given of a hexagonal lattice, with all its symmetry groups expanded in accordance with the lattice basis. On it are 5 distinct *poles.* And in fact, two of them correspond to D3—but split into the unimodularly inequivalent forms assumed by the same in the lattice-adapted system.

The visual difference between these two poles is that in one, the three axes pass through every 3-pole, while in the other, it passes only through the 6-poles—the number of poles it passes through is reduced down to a third. (While the metric ground form remains the same, one can see easily that with respect to the lattice basis, the commandment associated with D3 must take up different forms. In one, it will just be exchanging the basis vectors, which the other cannot be.)

<u>Chapter 4</u>

Time to recap.

We have found all the orthogonally inequivalent finite groups of homogeneous orthogonal transformations in two dimensions; given by Leonardo's infinite list.

We have found all the groups in the above list which can have invariant lattices; this is done by limiting the index n to 1, 2, 3, 4 or 6; we are left with a list of 10 symmetry groups.

Next, we derived from the second list the list of all unimodularly inequivalent (and homogenous) finite groups of linear transformations with integral coefficients; 10 splits to 13.

Finally, we include translations, and ask ourselves how many unimodularly inequivalent (non-homogeneous) discontinuous groups (with integral coefficients) with integral translations exist: The 13 goes to 17.

(Reminder: Virtually no rigorous proof was offered for most of these conclusions.)

In 1D, all the numbers would have been just 2. (Identity alone or identity and reflection.)

In 3D, the numbers are infinity, 32, 70 and 230.

The question may be extended to n-D spaces.

The indistinguishability of crystals with respect to certain transformations led to isotropies in physical properties. The following hypothesis was put forth: Whichever one of the 230 space groups our atomic lattice may belong to, it can contain no more than 3 linearly independent translations. This led to the law of rational indices in crystallography, says Weyl (a law relating the crystal to the metric).

The hypothesis just reformulates the fact that equivalent atoms form a regular point-set (under the given group).

The rotational part of the overall space group characterizes its macroscopic symmetry, while the space group itself "defines the microscopic symmetry hidden behind it".

A mention is made of how causality carries over symmetries from the conditions to the symptoms, culminating in the following grand suggestion:

"As far as I can see, all a priori statements in physics have their origin in symmetry."

It is reiterated that the *metric* form for a given lattice still remains a continuum, and the nature of the atom and other external factors gives us a most beautiful variety of these forms in nature. The force of this central duality in crystallography (discrete/continuous) is acknowledged in a comparison to the dichotomy of the genotype and the phenotype in biology…"but I will not deny that the general problem is in need of further epistemological clarification."

From crystallography, we move to the general application of the principle of symmetry in physics, and ultimately, mathematics.

The theory of relativity concerns itself with the geometric structure of space-time itself, preceding the study of symmetries of objects *in* said space-time.

In this space, Leibniz's idea of similarity is reiterated: Two things are similar in space if all "objective" statements about any one thing taken in itself are also true for the other taken in itself. But it is the notion of "objective" statements here which depends upon the aforementioned theory.

Now, Weyl says that physics has an "absolute standard length" tied into it (perhaps he refers to the Planck length?). Thus, two systems of reference differing spatially, both considering a given physical setup (which are equally admissible if, emulating Einstein, the physical laws take up the same algebraic expression), can be interconverted by the group of *physical automorphisms,* which must be congruent mappings (improper ones included, since left and right are indiscernible), due to the presence of the absolute standard length. These congruences—the physical automorphisms—can be expanded to include dilatations: The geometric automorphisms, dealing not with the relation of congruency in space itself but that of congruence between spatial figures.

But the congruences and the absolute standard mentioned above is restricted to space, whereas the theory of relativity unifies space-time. This "true" group of physical isomorphisms are called the *Lorentz group.* "I only want to point out that it is the inherent symmetry of the four-dimensional continuum of space and time that relativity deals with."

"Objectivity means invariance with respect to the group of automorphisms." And in the abstract sense, geometry consists of the investigation of invariants, given a group of transformations. The notion of symmetry corresponds with the finite subgroup under which the given object is invariant.

A ridiculously rough sketch of symmetry in quantum mechanics is given. I refrain from transcribing it.

The last leg: Symmetry in mathematics.

Consider a multivariable polynomial in rational coefficients, R(x1,x2,…xn), which goes to zero for the root-set E = {v1, v2,…vn}. The question is what permutations of the root-set can be made such that R still goes to 0. The set of these automorphisms form a *Galois group,* and we may speak of the Galois group of a usual polynomial f(x)=0, if its roots are v1, v2…vn. (This is over the field C.)

"Galois theory is nothing but relativity theory for the set E."

A concrete example: Consider $f(x)=x^2-2$. We obtain its roots. Weyl shows that any R such that R(v1,v2) is 0 must also satisfy R(v2,v1)=0. Thus, the Galois group consists of the identity and the transposition v2→v1, v1→v2. Most incredibly, *this fact is equivalent to the irrationality of the square root of 2!* This must be the much-hailed connection between group theory and field theory which Galois unearthed.

Another example from Galois theory: Consider the equation $z^p-1=0$. Its roots form the vertices of a regular p-sided polygon; furthermore, z=1 is a root, and $f(z)=(z-1)(z^{p-1}+\ldots z+1)=0$. Thus, the remaining roots are the roots of $z^{p-1}+\ldots z+1=0$.

If p is prime, these p-1 roots are "algebraically indiscernible", and the group of automorphisms for them is a cyclic group of order p-1.

This is illustrated diagrammatically for p=17: The group is now most evidently C16, which has subgroup C8, which has subgroup C4, which has C2, which has C1. Thus, we can also determine the roots of the equation in question using 4 consecutive equations of degree 2. But since the quadratic's roots can be extracted geometrically using just a ruler and a compass, the 3-gon, 5-gon, 17-gon are what can be constructed by just ruler and compass.

This can happen only when p is a prime number and p-1 is a power of 2; the next instance of this coincidence occurs for the number 257.

A final perspective on Galois theory: Consider the vector space (a+bsqrt2) over the field of rationals. What are its automorphisms? They are deduced to be nothing but the identity and the reflection (a,b)→(a,-b). (Note: The automorphism group of a vector space is just the set of all isomorphisms from V to V.)

It seems that Weyl uses the word 'field' instead of vector space, however. In fact, the above vector space is also a field, since it satisfies the field axioms as well, so there is no problem here.

A field, says Weyl, is an entity endowed with a structure; "space is another example of an entity endowed with a structure. What we have learnt from our whole discussion and what has indeed become a guiding principle in modern mathematics is this lesson: *Whenever you have to do with a structure-endowed entity, try to determine its group of automorphisms.* You can expect to gain a deep insight into (its) constitution in this way."

One can also study the structure of the group itself, since it is also a structure-endowed entity itself.

"Here the dog bites into its own tail, and maybe that is a clear enough warning for us to stop."

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[i] Weyl, H. (2015). Symmetry. Princeton: Princeton University Press.