Studies in geometric & Berezin quantization

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by

Aditya Dwarkesh
(19MS047)

5^{th} Year BS-MS
Department of Mathematics & Statistics
Indian Institute of Science Education and Research—Kolkata, India

Supervised by

Dr. Rukmini Dey

International Center for Theoretical Sciences, Bangalore

Co-supervised by

Dr. Soumalya Joardar

Indian Institute of Science Education and Research—Kolkata, India

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Declaration

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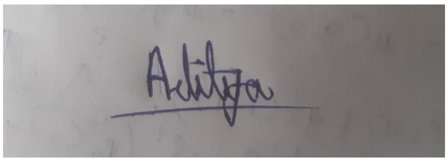

Aditya Dwarkesh
Department of Mathematics & Statistics
Indian Institute of Science Education and Research Kolkata
Mohanpur 741246, West Bengal, India

Certificate

Date: 13th May 2024

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Rukmini Dey
Associate Professor
International Center for
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Aditya

Contents

Abstract

Quantization is the attempt to describe a mathematically rigorous mode of approaching a quantum theory corresponding to a given classical theory. It is a problematic which lies at the crossroads of mathematics and physics, requiring both the unyielding rigour of the former and heuristic intuition from the latter. In this report, after establishing some geometric background, we discuss in some detail two ways of tackling this problem: Geometric quantization and Berezin quantization.

1 Introduction

Given a classical system, is there a natural way to map it to a corresponding quantum mechanical system?
This is the motivating question behind the mathematical theory of quantization. But before we proceed, everything in that sentence clamours for clarification.

- **Classical system:** Canonically, a particle in a classical system is described by position-momentum coordinates and a Lagrangian function; these, together with a set of well-established dynamical laws (codified by the Euler-Lagrange equations), allow one to predict the position-momentum coordinates of the particle at any point of time in the future.
  Generalizing this from $(\mathbb{R}^6)^n$ takes us to a smooth *manifold*, with some additional structure encoding dynamics. The right kind of additional structure turns out to be a closed, non-degenerate differential two-form known as a *symplectic form*; and classical observables are understood as smooth functions $f : M \to \mathbb{R}$.
  The study of manifolds with this additional structure is known as *symplectic geometry*. This having turned out to be the natural mathematical language for describing classical systems, we will begin with a brief overview of the same.

- **Quantum system:** Shortly after the inception of quantum theory, it was understood that quantum states are best understood as vectors (or, rather, *rays*) in a Hilbert space, and observables as Hermitian operators acting on this Hilbert space. This, therefore, is what yields a mathematical 'home' for quantum theory.

- **Natural:** There are certain heuristic requirements one would like such a map to obey. A set of such requirements was first articulated by Dirac [5]. We will discuss these quantization conditions (and others) in greater detail in chapter 3; and as we will see, these impose some serious constraints on the possibility of quantization maps.

- **Map:** Optimistically, one would imagine the map to, perhaps, be a *functor* $F : \mathbf{Symp} \to \mathbf{Hilb}$, where **Symp** is the category of symplectic manifolds (with morphisms being symplectomorphisms), and **Hilb** is the category of Hilbert spaces (with morphisms being Hermitian operators).
  Work towards solving formulating and sovling a category-theoretic version of the quantization problem was initiated by A. Weinstein [1]. It remains an open area of research, with much to be understood.

In the second chapter, after our discussion on symplectic geometry, we will establish some more preliminary notions required to understand the theory of geometric quantization. We discuss some basic structures on vector bundles and complex manifolds, and include a detour into the notion of a coadjoint orbit.

We discuss the theory proper in the third chapter. After first establishing the basic prequantization map, we go further and introduce the scheme of polarizations in order to make the Hilbert space suitably irreducible.
After this, two addendums to the basic theory are considered:

- **Obstructions:** A number of no-go theorems established by M. Gotay et. al. [8], proving the impossibility of quantization maps for certain spaces, shall be discussed. These also pave the way for some interesting questions to be formulated, which we shall state.

- **Quantization of states:** Traditionally, the theory of geometric quantization dealt only with the quantization of *observables* by taking smooth maps to Hermitian operators. However, A. Odzijewicz [16] extended this setup to the quantization of *states* as well. We will briefly go over this extension.

In the fourth chapter, we discuss an alternative scheme for quantization; namely, *deformation quantization*, as initiated by M. Flato et. al. [2]. We will begin by discussing an important deformation map on the Poisson algebra of classical observables, first established by Fedosov [6], move onto another star product defined by Berezin [3], and end by tying this back concretely into the theory of geometric quantization through a result proven by Rawnsley et. al. [17].

2

Geometric preliminaries

This chapter is primarily based on [18] and [14].

2.1 Symplectic geometry

2.1.1 Basic concepts

We introduce the notion of a symplectic manifold and establish some basic properties, ultimately tying it into classical mechanics.

Definition 2.1.1 (Symplectic manifold)**.** A **symplectic manifold** is a pair (M, ω), where M is a smooth manifold and ω is a 2-form on M such that:

1. $d\omega = 0$ (ω is closed)
2. The map $T_pM \to T_p^*M, X \mapsto i_X\omega$ is a linear isomorphism at each $p \in M$ (ω is nondegenerate).

Definition 2.1.2 (Symplectic vector space)**.** A **symplectic vector space** (V, ω) is a vector space V and an antisymmetric, nondegenerate bilinear form ω on V.

A symplectic vector space is a canonical example of a symplectic manifold. The tangent space of any symplectic manifold is, in turn, an example of a symplectic vector space.
For any given subspace of a symplectic vector space $F \subseteq V$, we may speak of its *symplectic complement*, defined by $F^\perp = \{X : \omega(X, Y) = 0 \forall Y \in F\}$.

We now introduce some terminology. A subspace $F \subseteq V$ of a symplectic vector space is said to be:

- *Isotropic* if $F \subseteq F^{\perp}$
- *Coisotropic* if $F^{\perp} \subseteq F$
- *Symplectic* if $F \cap F^{\perp} = \{0\}$
- *Lagrangian* if $F = F^{\perp}$.

Note that the categories are not exhaustive.

Lemma 2.1.1. *Every finite-dimensional symplectic vector space has even dimension and contains a Lagrangian subspace.[18]*

Theorem 2.1.2. *Let (V, ω) be a 2n-dimensional symplectic vector space. Then, V has a basis $\{X^1, ...X^n, Y_1, ...Y_n\}$ such that:*

- $\omega(X^a, X^b) = 0$
- $2\omega(X^a, Y_b) = \delta^a_b$
- $\omega(Y_a, Y_b) = 0$.

*Such a basis is called a **symplectic frame**.[18]*

Example 2.1.1 (Cotangent bundle). *The cotangent bundle T^*M of a smooth manifold M is an important example of a symplectic manifold. The relevant 2-form is $\omega = dp_a \wedge dq^a$, where $\{q^i\}$ are the components of the covectors and $\{p_i\}$ are the coordinates on M.*
It is possible to show that ω is globally determined in a coordinate-independent manner as $d\theta$, where θ is a one-form on T^*M defined by $(q, p) \mapsto (X \mapsto q(\pi_* X))$, where, for $m = (q, p), X \in T_m(T^*M)$, and π_* is the pushforward of the projection. In terms of coordinates, $\theta = q_a dp^a$.
As it turns out, all symplectic manifolds have this form (locally). This will be the content of the next theorem.
Some work needs to be done to see how the coordinate-independent definition is actually the same as the one in terms of coordinates.
Let $X \in \chi(T^*M) = \sum f_i \frac{\partial}{\partial q_i} + \sum g_i \frac{\partial}{\partial p_i}$. Now, $(\pi_* X_m)(f) = X(f \circ \pi)$ (where $f \in C^{\infty}(M)$) $= g_i(m) \frac{\partial f}{\partial p_i}|_p$ (because $f \circ \pi$ does not depend on $\{q_i\}$).
In other words, $\pi_* X = \sum g_i \frac{\partial}{\partial p_i}$. Then, $q(\pi_* X_m) = \sum g_i(p) q(\frac{\partial}{\partial p_i}|_p)$ (where $m = (q, p)$).
On the other hand, $(q_a dp^a)(\sum f_i \frac{\partial}{\partial q_i} + \sum g_i \frac{\partial}{\partial p_i})(q, p) = \sum g_i(p) q(\frac{\partial}{\partial p_i}|_p)$ directly.

Discussion 2.1.1. *Some new concepts will aid us in understanding the subsequent results.*

- *Time-dependent vector fields: Let* X *be a vector field such that* $X = X^a(x,t)\frac{\partial}{\partial x^a}$*, so that it is a map* $M \times \mathbb{R} \to TM$ *such that* $X(m,t) \in T_m M$*. Vector fields with such an additional parameter* t *are called time-dependent vector fields.*
  The 'ordinary vector field' $\tilde{X} = X + T$ *is defined by* $\tilde{X} = X^a \frac{\partial}{\partial x^a} + \frac{\partial}{\partial t}$*. We will have* $X \in \chi(M \times \mathbb{R})$*.*
  Finally, we define the time derivative of $X, \partial_t X = \partial_t X^a \frac{\partial}{\partial x^a}$*.*

- *Integral curves: Let* $\gamma(t) = (x^1 \circ \gamma(t), ..., x^n \circ \gamma(t))$*, where* $x : M \to \mathbb{R}^n$ *is a chart, and* $\gamma : I \to M$ *is such that* $\frac{d\gamma^a}{dt} = X^a(\gamma(t))$*. Then,* $t \mapsto \gamma(t)$ *is said to be an integral curve of the time-independent vector field* X*.*
  For a time-dependent vector field, the curve is defined by $\frac{d\gamma^a}{dt} = X^a(\gamma(t), t)$*.*

- *Flow: Let* X *be a time-dependent vector field and* γ *be an integral curve such that* $\gamma(t) = m$*. Then, the flow of* X *is the diffeomorphism* $\rho_{tt'} : m \mapsto \gamma(t')$*. Characterized by the two subscripted parameters, this map takes the image of* γ *on the first of them and returns its image on the second.*

- *Time-dependent differential forms: A time-dependent* p*-form is defined as a map* $\alpha : M \times \mathbb{R} \to \Lambda^p(TM)$ *such that* $\alpha_{(m,t)} \in \Lambda^p(T_m M)$*.*

- *Lie derivative: The Lie derivative of a differential form along a vector field (both being time-dependent)* $\mathscr{L}_X \alpha := i_X(d\alpha) + d(i_X \alpha) + \partial_t \alpha$*.*

Lemma 2.1.3. *Let* X *be a time-dependent vector field,* $\rho_{tt'}$ *its flow, and* α *a time-dependent differential form. Then, the following holds:*

$$\frac{d}{dt}_{t=t_1}(\rho^*_{tt_0}\tau_t)_p = (\rho^*_{t_1 t_0}(\mathscr{L}_X \tau)_{t=t_1})_p$$

Lemma 2.1.4. *Let* ω, ω' *be symplectic structures on* M *and* $m \in M$*. If* $\omega(m) = \omega'(m)$*, there exist neighbourhoods* U, V *of* m *and a diffeomorphism* $\rho : U \to V$ *such that* $\rho(m) = m, \rho^*(\omega') = \omega$*.*

Proof. Since $\omega' - \omega$ is closed, by the Poincaré lemma, there will exist a 1-form α in some neighbourhood W of m such that $d\alpha = \omega' - \omega$. If necessary, add a closed 1-form to α to ensure that $\alpha(m) = 0$.
Next, let $\Omega = \omega + t(\omega' - \omega)$. Note that this is another closed 2-form, with $\Omega(m) = \omega(m)$. Let X be the vector field such that $i_X \Omega + \alpha = 0$.

- X is well-defined: Firstly, note that since $\omega(m) = \Omega(m)$ and ω is non-degenerate (by virtue of being a symplectic form), Ω will also be non-degenerate in a neighbourhood of m (using the fact that the determinant function is continuous).
  But this means that $X \mapsto i_X \Omega$ is a linear isomorphism in that neighbourhood. The claim follows from this.

Note that we will have $X(m) = 0$, because $\alpha(m) = 0 \implies \Omega_m(X(m), Y(m)) = 0$ for all $Y \in \chi(M) \implies X(m) = 0$ (by non-degeneracy of Ω).
Let $\rho_{tt'}$ be the flow of X. Then, we claim that $\rho := \rho_{01}$ is the required diffeomorphism.

- $\rho^*_{tt'}\Omega(t') = \Omega(t)$: First, let us compute $\mathscr{L}_X\Omega = i_X(d\Omega) + d(i_X\Omega) + \partial_t\Omega = 0 - d\alpha + \omega - \omega'$ (since $d\Omega = 0$) $= 0$. This means that $\frac{d}{dt}_{t=t''}(\rho^*_{tt'}\Omega_t) = 0 \implies \rho^*_{tt'}\Omega(t) = \rho^*_{t't'}\Omega(t') = \Omega(t')$.

But now, since $\Omega(1) = \omega', \Omega(0) = \omega$, it follows that $\rho^*\omega' = \omega$.

- $\rho(m) = m$: Recall that $X(m) = 0$. This means that the integral curve γ of X at m is constant; the claim follows from this.

This shows that the diffeomorphism ρ is of the required type, and we are done. □

Theorem 2.1.5 (Darboux)**.** *Let (M, ω) be a 2n-dimensional symplectic manifold, and $m \in M$. There is a neighbourhood U of m and a coordinate system $\{p_a, q^b\}$ $(a, b = 1, 2, ..., n)$ on U such that $\omega = dp_a \wedge dq^a$ on U.*

Proof. Consider $\omega_m \in \Lambda^2(T_mM)$. By theorem 2.1.2, we can find a symplectic frame for T_mM; let us write ω'_m for the representation of the two-form in this basis. Then, necessarily $\omega' = dr_a \wedge ds^a$.
By the above lemma, there exists a diffeomorphism ρ defined on a neighbourhood U of m such that $\rho^*(\omega') = \omega$. Define $p_a = r_a \circ \rho; q^a = s^a \circ \rho$. Then, $\omega = dp_a \wedge dq^a$ in U, and we are done. □

These coordinates are called *canonical* or *Darboux* coordinates. Thus, in some sense, all symplectic forms look the 'same' locally. This can be contrasted with the situation on a Riemannian manifold, wherein metrics have various local invariants.
Thus, we can also, in the general case, find in the neighbourhood of each point a 1-form θ such that $\omega = d\theta$. Such a 1-form is called a *symplectic potential.*

2.1.2 Hamiltonian vector fields

Definition 2.1.3. Let (M, ω) be a symplectic manifold and, given an $f \in C^\infty(M)$, let X_f be the vector field such that $i_{X_f}(\omega) + df = 0$.
We shall call X_f the **Hamiltonian vector field**, and its flow ρ_t the **canonical flow**, generated by f.

Discussion 2.1.2. *In local coordinates, we can work out $X_f = \frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial f}{\partial q_a}\frac{\partial}{\partial p^a}$ (using Darboux's theorem).*
On the physics side of things, we often intend the symplectic manifold to represent the phase space of a classical system; $C^\infty(M)$ functions, in turn, are classical observables.

- f : *On the one hand, a classical observable is a measurable quantity which takes on a value for a given state of the system.*

- ρ_t : *On the other, it generates a one-parameter family of canonical transformations.*

The vector field X_f encodes the geometric connection between these two roles of the observable. Since $\mathscr{L}_{X_f}\omega = 0$, the flow of X_f, ρ_t, will in fact always satisfy $\rho_t^*(\omega) = \omega$ (via an argument similar to the one used in the proof of lemma 2.1.4), making it a (local) canonical diffeomorphism.

Definition 2.1.4 (Poisson bracket)**.** The **Poisson bracket** of $f, g \in C^\infty(M)$ is $\{f, g\} := X_f(g)$.
In canonical coordinates, $\{f, g\} = \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a} - \frac{\partial g}{\partial p_a}\frac{\partial f}{\partial q^a}$.

Let us denote by $V^H(M)$ the set of all Hamiltonian vector fields. $(C^\infty(M), \{,\})$ is an infinite-dimensional Lie algebra. The map $f \mapsto X_f$ is (surjective) a Lie algebra homomorphism [13] $C^\infty(M) \to V^H(M)$, with kernel equal to the set of constant functions. So, by the first isomorphism theorem, $C^\infty(M)/\mathbb{R} \cong V^H(M)$.

Discussion 2.1.3 (Noether's theorem)**.** *Establishing these notions allows us to approach the mathematical kernel of Noether's theorem, which famously relates the conserved quantities in a system with its symmetries:*
If, for $f, g \in C^\infty(M), \{f, g\} = 0$, then $g \circ \gamma_f : I \to \mathbb{R}$ is a constant; where γ_f is an integral curve of X_f.
This is easy to verify by writing everything out in canonical coordinates.

2.1.3 Symplectic reduction

Often, one has to deal with manifolds with a closed two-forms which has certain degeneracies. This may happen, for example, in constrained systems. The procedure in this section describes how to recover a genuine symplectic manifold from that setup. But before that, introducing some new notions will prove useful.

Discussion 2.1.4. *Let M be a smooth manifold.*

- *Distribution: A subbundle P of the tangent bundle TM such that P_m is a subspace of T_mM and varies smoothly with m.*

- *Regular: A distribution P is called **regular** if each subspace P_m has the same dimension.*

- *Transverse: A submanifold $\Sigma \subset M$ is **transverse** to P if $T_m\Sigma + P_m = T_mM$ at each $m \in M$.*

- *Integral manifold: An immersed submanifold $\Lambda \subset M$ is an* ***integral manifold*** *of a distribution P if $P_m = T_m\Lambda$ for all $m \in \Lambda$.*

- *Integrable: A distribution is* ***integrable*** *(or involutive) if $X, Y \in V_P(M) \implies [X, Y] \in V_P(M)$.*

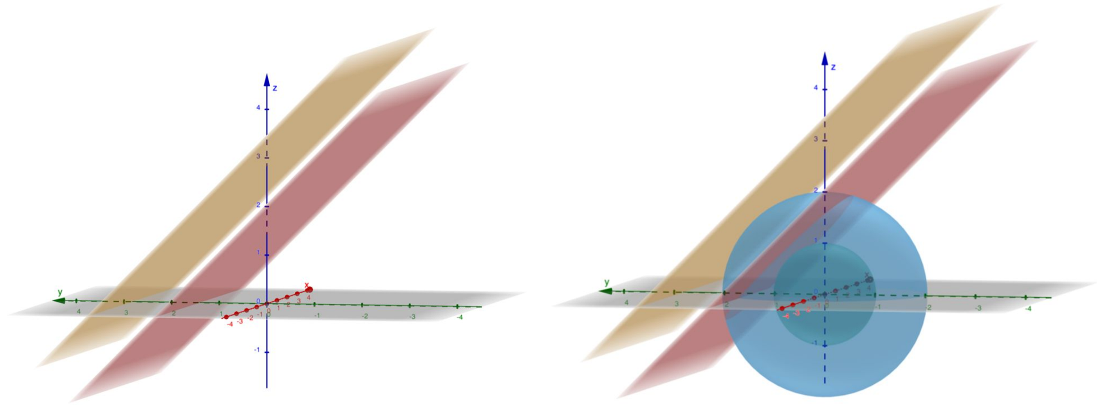
- *Foliation: An integrable distribution is called a* ***foliation****.*

- *Leaves: Let P be a foliation on M, and $\Lambda \subset M$ be a (connected) integral manifold of P such that there is no $\Lambda \subset \Lambda' \subset M$ such that Λ' is an integral manifold of P. We call such Λ* ***leaves****.*

- *Reducible: Let P be a foliation on M. We define the following equivalence relation on $M : p \sim q \iff p, q \in \Lambda$ for some leaf Λ of P. Now, if the quotient space $M/\sim$ is a Hausdorff manifold, we call P* ***reducible*** *and $M/\sim$ the space of leaves of $P, M/P$.*

The Frobenius theorem states that a distribution is a foliation iff it has an integral manifold. [12]

Example 2.1.2. *It will be instructive to offer a simple example of a reducible distribution before moving on. Let $M = \mathbb{R}^3$, and consider the following:*



(a) A distribution on $\mathbb{R}^3$ (b) The leaves of the distribution

It is now evident that if we pass down to the space of leaves of this foliation, we will be left with $[0, \infty)$. Since this is Hausdorff, we conclude that the distribution was reducible.

Definition 2.1.5 (Characteristic distribution)**.** Let C be a smooth manifold and σ be a two-form on C. Let $K_p := \{X_p \in T_pM | i_{X_p}(\sigma_p) = 0\}$. Suppose further that K is regular. Then, we call K the **characteristic distribution** of σ.

This distribution collects all the problematic, degeneracy-causing vector fields together. We want, in some appropriate sense, to quotient it out.

Lemma 2.1.6. *If σ is closed, K is a foliation.*

Proof. Let $X, Y \in V_K(M)$. We need to show that $i_{[X,Y]}(\sigma) = 0$. But this follows from the fact that $3d\sigma(X, Y, Z) = -\sigma([X, Y], Z)$. □

If this condition holds, we call K the *characteristic foliation.*

Definition 2.1.6 (Presymplectic manifold)**.** We call (C, σ) a **presymplectic manifold** if σ generates a regular distribution and is closed. Furthermore, we call a presymplectic manifold **reducible** if its characteristic foliation is reducible.

We call a closed regular two-form on a manifold a *presymplectic form.* The nomenclature is suggestive because it generates a genuine symplectic manifold in the following manner.

Definition 2.1.7 (Reduced phase space)**.** Let (C, σ) be a reducible presymplectic manifold. Let M' be the space of leaves, and ω' be a two-form on M' defined by being such that $\sigma = q^*(\omega'), q$ being the quotient map. Then, (M', ω') is a symplectic manifold, called the **reduced phase space** of (C, σ).

Well-definedness of ω' well-defined follows from the fact that q is a submersion; and so its pullback is injective; and losedness from the fact that d commutes with the pullback.

Example 2.1.3. *Let* $C = \mathbb{R}^3/\{0\}, \sigma = \frac{1}{r^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$*. We show that* (C, σ) *is a presymplectic manifold and compute its reduced phase space* (M, ω)*.*

Definition 2.1.8. We call a submanifold C of a symplectic manifold (M, ω) **isotropic/coisotropic/ Lagrangian/symplectic** if its tangent space is of the corresponding type as a subspace of T_pM for every $p \in C$.

Discussion 2.1.5. *One is often interested in performing symplectic reduction when one has, due to some constraint on the phase space, a submanifold* C *of* (M, ω) *such that* $\sigma = \omega|_C$ *is regular. The relevant distribution is defined by* $K_p = T_pC \cap T_pC^\perp$*.*
In the symplectic case, reduction is trivial, because the distribution vanishes identically. In the isotropic and Lagrangian case, the distribution is the whole tangent bundle; and so the reduced phase space itself becomes trivial.
That leaves us with the coisotropic case for something interesting to happen. The following result holds:

$$K_p = \text{span}\{(X_f)_p : f|_C = \text{constant}\}$$

2.1.4 Coadjoint orbits

Definition 2.1.9 (Symplectic action)**.** Let (M, ω) be a symplectic manifold and G be a Lie group acting on it. We say that the action of G is **symplectic** if $f_g : x \mapsto g \cdot x$ is a symplectomorphism for all $g \in G$.

Definition 2.1.10 (Adjoint action)**.** Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. The derivative at e of the conjugacy map $G \to G, a \mapsto gag^{-1}$ is known as the **adjoint action** of G on $\mathfrak{g}, Ad_g : \mathfrak{g} \to \mathfrak{g}$.

Definition 2.1.11 (Coadjoint action)**.** Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and Ad_g the adjoint action. Let $\mathfrak{g}^*$ be the vector space dual of $\mathfrak{g}$. The **coadjoint action** $Ad_g^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is just the dual map of the adjoint action, defined by:

$$Ad_g^*(l) = l(Ad_g)$$

for $l \in \mathfrak{g}^*$.

The map $g \mapsto Ad_g, G \to \mathrm{Aut}(\mathfrak{g})$ is known as the *adjoint representation* of the Lie group; the map $g \mapsto Ad_g^*, G \to \mathrm{Aut}(\mathfrak{g}^*)$ is known as the *coadjoint representation* of the Lie group.

Definition 2.1.12 (Infinitesimal action)**.** Let G be a Lie group acting on a manifold M. Its **infinitesimal action** is the vector field defined by $\rho(\zeta)_x := d[\exp(t\zeta) \cdot x]_0$ for $\zeta \in \mathfrak{g}$ (wherein we look at $\exp(t\zeta) \cdot x : I \to M$).

Recall that $\exp : \mathfrak{g} \to G$ was defined as $X \mapsto \gamma(1)$, where $\gamma : I \to G$ was a curve such that $\gamma'(0) = X$.

Definition 2.1.13 (Momentum map)**.** A **momentum map** for a symplectic G-action on a symplectic manifold (M, ω) is a map $\mu : M \to \mathfrak{g}^*$ such that the following hold:

1. $d[\mu(x)(\zeta)] = i_{\rho(\zeta)}\omega$
2. $\mu(g \cdot x) = Ad_g^*(\mu(x))$.

Here, we look at $\mu(x)(\zeta) : M \to \mathbb{R}$.

A symplectic Lie group action on a symplectic manifold is called a **Hamiltonian group action** if a momentum map μ exists, and (M, ω, G, μ) is called a Hamiltonian $G-$space.

Discussion 2.1.6. *There is a fair bit of physical context behind the momentum map, which it will be useful to make transparent.*
Firstly, note that one can also consider the so-called ***comomentum map****,* $\mu^* : \mathfrak{g} \to C^\infty(M)$*, which is defined as* $\mu^*(\zeta)(x) := \mu(x)(\zeta)$*.*
It can be checked that the two properties stated in the above definition are now equivalent to:

1. $\mu^*(\zeta) \in C^\infty(M)$ *is a Hamiltonian function for* $\rho(\zeta)$*.*
2. μ^* *is a Lie algebra homomorphism.*

The upshot of this is that we have found a natural way to assign to each $\zeta \in \mathfrak{g}$ *an observable* $\mu^*(\zeta) \in C^\infty(M)$*. This assignment is what allows us to extract various symmetries from our system.*
The prototypical example is when $\mathfrak{g}$ *is the Lie algebra of the rotation group, in which case the associated observables are just the components of the angular momentum.*

Definition 2.1.14 (Coadjoint orbit)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}^*$. The **coadjoint orbit** of $f \in \mathfrak{g}^*$ is defined as $\Theta_f = \{h \in \mathfrak{g}^* | h = Ad^*_g f, g \in G\}$.
The **isotropy group** of $f \in \mathfrak{g}^*$ is defined as $H = \{g \in G | Ad^*_g f = f\}$.

We have an isomorphism $\Theta_f \cong G/H$ ($\implies T_p\Theta_f \cong \frac{\mathfrak{g}}{\mathfrak{h}}$). In fact, not only is it a smooth manifold, each coadjoint orbit comes equipped with a natural symplectic structure.

Lemma 2.1.7. *Let G be a Lie group, and X_A denote the infinitesimal action associated with $A \in \mathfrak{g}$. Then, $\omega(X_A, X_B) := \frac{1}{2} f([A, B])$ is a symplectic form on Θ_f.*

Discussion 2.1.7. *The fact that $SU(n+1)$, with the obvious action, acts transitively on $\mathbb{P}^n$ suggests that the latter can be realized as a coadjoint orbit of the former. Let us see how this comes to be. Our job is, to begin with:*

- *Finding an element $f \in \mathfrak{g}^*$ such that the orbit of f is $\mathbb{P}^n$. But since we know $\mathbb{P}^n \cong SU(n+1)/S(U(1) \times U(n))$ and $\Theta_f \cong G/H$, this amounts to:*

- *Finding an element $f \in \mathfrak{g}^*$ whose isotropy group is $H = S(U(1) \times U(n))$.*

The question becomes how we may hunt down this privileged element of the dual Lie algebra. We know that coadjoint orbits come with a symplectic structure; the fact that $\mathbb{P}^n$ also has one (the Fubini-Study metric, which we will see in further detail later) should suggest that it must be used somewhere in this hunt. But first, it will prove illuminating to unpack the structure of this f a bit more.

- *Some quick computation can show that the previous statement is equivalent to having $f(B) = f(gBg^{-1})$ for $g \in H, B \in \mathfrak{g}$.*

- *Since H is connected and compact, the exponential map is surjective and we may write $g = \exp(tA)$ for some $A \in \mathfrak{h}$.*

- *Again, some quick computation can show that $df(e^{tA}Be^{-tA}) = f([A, B])$.*

This is very nearly the structure of the symplectic form given by lemma 2.1.7! Suppose we set $F([A, B]) = 2\omega_{FS}(X_A, X_B)$. Let us fix some $B \in \mathfrak{g}$; then, F is a genuine element of $\mathfrak{g}^*$.

- *ω_{FS} is defined on $\mathbb{P}^n = G/H \implies \omega_{FS}$ is degenerate exactly along $H \implies F([A, B]) = 0$ only for $A \in \mathfrak{h}$.*

But this last statement is equivalent to having $dF(e^{tA}Be^{-tA}) = 0$ for precisely $A \in \mathfrak{h}$; which, in turn (running back up to the first statement) means $F(B) = F(gBg^{-1})$ for $g \in H$. This shows that F is indeed the linear functional required of us.
Note that ω_{FS} being nondegenerate elsewhere was critical in ensuring that the isotropy group of F is not larger than H.

2.2 Vector bundles

2.2.1 Basic concepts

Definition 2.2.1 (Vector bundle)**.** Let M be a smooth manifold. A **vector bundle** over M is a smooth manifold V, together with a smooth map $\pi : V \to M$ such that the following hold:

1. For each $m \in M, V_m = \pi^{-1}(m)$ is an n-dimensional vector space over $\mathbb{R}$.

2. M has an open cover (U_i, τ_i), wherein $\tau_i : U_i \times \mathbb{R}^n \to \pi^{-1}(U_i) \subseteq V$ is a diffeomorphism which becomes a linear map $\mathbb{R}^n \to V_m$ on restriction to $\{m\} \times \mathbb{R}^n$ for each $m \in U_i$.

The dimension n of the fibres is to be constant throughout M. It is known as the *rank* of the bundle.
A pair (U_i, τ_i) satisfying the condition described above is known as a *local trivialization.* (The nomenclature is supposed to suggest that the map π locally looks like the projection $U \times \mathbb{R}^n \to U$—since, through τ_i, the inverse image of a singleton $\pi^{-1}(m)$ ends up being $\mathbb{R}^n$.)
A vector bundle whose fibre is one dimensional is called a *line bundle.*
The composite functions $\tau_i^{-1} \circ \tau_j : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$ can be written as $(x, v) \mapsto (x, g_{U_iU_j}(x)v)$, where $g_{U_iU_j}$ are the *transition functions* of the vector bundle and map to $\mathrm{GL}(k)$.

Example 2.2.1. *The canonical example of a vector bundle is the tangent bundle TM, with the projection map $\pi : TM \to M$. Each fibre is $T_pM \cong \mathbb{R}^n$ (assuming the manifold dimension is n). The transition functions will be the Jacobian of the coordinate transformations.*

Definition 2.2.2 (Section)**.** A map $s : U \subseteq M \to V$ such that $\pi \circ s = Id_M$ is called a **section** over U.

We shall denote the space of smooth sections over U by $C_V^\infty(U)$. Note that $\tau_i(-, x)$ is a section. Sections are the relevant generalizations of vector fields in this setup (the target space goes from TM to any V).
Given a local trivialization (U_i, τ_i), we shall call the section $s_i = \tau_i(\cdot, 1)$ its *unit section.* (By linearity, this will actually determine the trivialization.) Note that $s_i \in C_V^\infty(U_i)$.

Definition 2.2.3 (Transition functions)**.** Let $(U_i, \tau_i), (U_j, \tau_j)$ be two local trivializations of a line bundle $L \to M$ with sections s_i, s_j. The **transition function** $c_{ij} \in C^\infty(U_i \cap U_j)$ is the function such that $s_j = c_{ij}s_i$.

Lemma 2.2.1. *Let M be a smooth manifold with a contractible open cover $\{U_i\}$ and a collection of functions $\{c_{ij}\}$.*
Then, if the following ***cocyle relations*** *hold, it is possible to reconstruct a line bundle on it with transition functions $\{c_{ij}\}$ (up to a normalizing constant):*

- $c_{jk} = c_{kj}^{-1}$
- $c_{ij}c_{jk}c_{ki} = 1$ [18]

Example 2.2.2 (Hyperplane bundle)**.** *The following line bundle over $\mathbb{CP}^n$ will be used later on. It is natural to see this as the inverse or dual of the universal bundle. The latter structure is what we shall describe explicitly.*
Let $L := U_{1,n}$ be the disjoint union of all one-dimensional subspaces of $\mathbb{C}^n$. To specify an element in it, we need a line and a point (in that line). Therefore, it has a natural realization as a subset of $G_{1,n} \times \mathbb{C}^n$, from which it can inherit a smooth structure.
Define $\pi : U_{1,n} \to \mathbb{CP}^n, t(z_1, ..., z_n) \mapsto [z_1, ..., z_n] \forall t \in \mathbb{C}$. Clearly, the fibres are each $\mathbb{C}$ for such a map.
Our local trivializations are (U_α, h_α), where $\{U_\alpha\}$ is the standard open cover on projective space, and $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}, t(z_1, ..., z_n) \to ([z_1, ..., z_n], tz_\alpha)$.
Finally, it is easy to compute that the transition functions (on, say, $U_\alpha \cap U_\beta$) will take the form $g_{\alpha\beta} := \frac{z_\alpha}{z_\beta}$.
This is the universal line bundle, which we may denote by L. The hyperplane bundle is merely its inverse $H = L^{-1}$, with transition functions given by $g_{\alpha\beta} := \frac{z_\beta}{z_\alpha}$. The open cover remains the same under the inversion operation.

2.2.2 Connections

Definition 2.2.4 (Connections)**.** Let V be a vector bundle over M. A **connection** on V is an operator ∇ which assigns to any $s \in C_V^\infty(M)$ a map $\nabla s : TM \to V$ such that $(\nabla s)_p : T_pM \to V_p$ is linear, ∇ distributes over addition and satisfies $\nabla(fs) = (df)s + f\nabla s$ for any function $f \in C^\infty(M)$.

Discussion 2.2.1. *Let $L \to M$ be a complex line bundle (in other words, replace $\mathbb{R}$ with $\mathbb{C}$ everywhere), and ∇ be a connection on L. Further, let (U, τ) be a local trivialization; and s be its unit section.*
The potential 1-form $\Theta \in \Omega^1(U)$ is implicitly defined (for a given connection and trivialization) by $\nabla s = -i\Theta s$. (Unpacked, this means that, for any vector field X on U and $p \in U$, $(\nabla_X s)(p) = -i[\Theta(X)](p) \cdot s(p) \in V_p$.)
The ***curvature*** *of a connection on a line bundle is defined as the two-form $\Omega = d\Theta$.*

Lemma 2.2.2. $\Omega(X, Y) = i([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})$

Proof. Let s' be any smooth section, and $\psi \in C^\infty(M)$ be the function which makes $s' = \psi s$, where s is the unit section.
Then, $\nabla_X s' = \nabla_X(\psi s) = (X(\psi) - i\psi\Theta(X))s$, by the Leibniz rule.
We now apply this repeatedly on the right-hand side above. This yields, after some simplification,

$$i(\Theta([X, Y]) - [\Theta(X), \Theta(Y)] + (Y(\Theta(X)) - (X(\Theta(Y))]s' = i'(d\Theta)s'$$

Since $\Omega = d\Theta$ (and $[\Theta(X), \Theta(Y)] = 0$), the formula follows as stated. □

If $\Omega = 0$, that is, the line bundle has a flat connection, we call it a *locally constant line bundle.*

Discussion 2.2.2. *There are two indirect ways of specifying connections which we shall now discuss.*

- *Connection potentials: Suppose we are given a collection of potential 1-forms* $\{\Theta_j\}$ *on the manifold. Under what patching condition can we reconstruct a connection from them?*

$$\Theta_k - \Theta_j = i\frac{dc_{jk}}{c_{jk}}$$

- *Connection forms: Instead of a collection of 1-forms on the manifold, we can also specify the connection with one 1-form on* $L^\times = L - L_0$ *(where* L_0 *is the submanifold of* L *which consists of all the zero vectors across the fibres).*

$$\alpha = \pi^*(\Theta) - i(\tau^{-1})^*(\frac{dz}{z})$$

  This equation may require some unpacking.
  First, since $\pi : L \to M, \pi^* : \Omega^*(M) \to \Omega^*(L)$*, and* $\pi^*(\Theta)$ *is a 1-form in some neighbourhood of* L*.*
  Next, $\tau : U \times \mathbb{C} \to \pi^{-1}(U)$*, making* $(\tau^{-1})^* : \Omega^*(U \times \mathbb{C}) \to \Omega^*(\pi^{-1}(U))$*.*
  Since we are working on a complex manifold, z *is just the local chart; so,* $\frac{dz}{z}$ *makes sense as a 1-form on* M*. One may imagine that the action of* $(\tau^{-1})^*$ *on the* $\Omega(\mathbb{C})$ *component is ignored.*
  It can be shown that this form is independent of local trivialization.
  Ultimately, the connection can be recovered by $\nabla s = -i(s^*\alpha)s$*, because* $s^*(\alpha) = (\pi \circ s)^*(\Theta) - i(\tau^{-1} \circ s)^*(\frac{dz}{z}) = \Theta$ *(the first is the pullback of the identity map; the second is the pullback of the constant map).*

2.2.3 Čech cohomology

We define a new cohomology theory on a manifold, called the Čech cohomology. Let M be a smooth manifold and $\mathcal{U} = \{U_i\}$ be an open cover of M.

Definition 2.2.5. A **p-cochain relative to** $\mathcal{U}$ is a collection of (typically, smooth) functions $\mathcal{F}_p = \{f_{i,j,\ldots k}\}$ (where the indices are from the same indexing set as of the open cover's) such that the following hold:

1. Each function $f_{i,j,\ldots k}$ has $p+1$ indices and is defined on $U_i \cap U_j \ldots \cap U_k$

2. $\mathcal{F}_p$ contains at least one function $f_{i,j\ldots k}$ for each ordered set of p+1 indices for which $U_i \cap U_j \ldots \cap U_k$ is non-empty

3. Each function is skew-symmetric under permutation of its indices

Definition 2.2.6. The **coboundary operator** $\delta : \mathscr{F}_p \to \mathscr{F}_{p+1}$ maps each $f_{j,k...l} \mapsto (p+2)\rho_i f_{j,k...l}$, where $\rho_i f_{j,k...l}$ is the restriction of $f_{j,k...l}$ to $U_i \cap U_j \cap U_k \cap ...U_l$.

Discussion 2.2.3. *We want to see that this defines a cohomology on the topological space. Firstly, $\delta^2 = 0$: This is because any function, when acted upon by the coboundary operator twice, will be both symmetric (switch the two restriction functions) as well as skew-symmetric (due to condition 3 in the definition of a p-cochain). Call functions in the kernel of δ cocycles, and ones in its range coboundaries. Both form abelian groups under pointwise operations.*
The p^{th} ***Čech cohomology group*** *is the p—cocycles modulo the $(p-1)$—cochains.*

Theorem 2.2.3. *Let M be a smooth manifold. Then, its de Rham cohomology is isomorphic to its Čech cohomology.*

Proof. We restrict ourselves to the case where $\mathscr{U}$ is locally finite and contractible, and the functions in each cochain are locally constant.
The isomorphism is the map $f \mapsto \alpha_f$, where f is a p—cochain and $\alpha_f := f_{i,j...k} h_i dh_j \wedge ... \wedge dh_k$, where $\{h_i\}$ is a partition of unity subordinate to $\mathscr{U}$, and there is a sum over the repeated indices.
We briefly indicate the proof of surjectivity for the case $n = 2$, because it will help us later: Let α be any closed two form. Locally, $\alpha = d\beta_i$. Locally, $\beta_i - \beta_j = dg_{ij}$.
It can be checked that the image of the 2—cochain $f = \{f_{ijk} = (g_{ij} + g_{jk} + g_{ki})\}$ is α. □

2.3 Complex geometry

Definition 2.3.1 (Almost complex manifold)**.** Let M be a smooth real manifold of even dimension. An **almost complex structure** J on M is a map $J : TM \to TM$ such that $J_p : T_pM \to T_pM$ is a linear map satisfying $J_p^2 = -1$ for all $p \in M$.
A manifold with an almost complex structure is an **almost complex manifold**.

Definition 2.3.2 (Complex manifold)**.** Let M be a second-countable Hausdorff topological space with an atlas $\{(U_i, \phi_i)\}$ such that $\phi_i : U_i \to U \subseteq \mathbb{C}^m$ is a homeomorphism. Furthermore, suppose all the transition maps $\psi_{ji} = \phi_j \circ \phi_i^{-1}$ are holomorphic. Then, M is a **complex manifold** with complex dimension m.

A complex manifold of dimension m can also be cast as a real manifold of dimension $2m$. A complex manifold will always have an almost complex structure. On the other hand, an almost complex manifold cannot, in general, be given a genuine complex structure. The following result tells us when this is possible.

Theorem 2.3.1 (Newlander-Nirenberg)**.** *Let M be a smooth real manifold with an almost complex structure J. Then, M can be given a complex structure if the distribution $P = \{X - iJ(X) | X \in \chi(M)\}$ is integrable.[15]*

Note that in the above, $J_p : T_pM \to T_pM$ must first be extended linearly to the complexification of T_pM. In the standard basis, J_p then takes the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Put another way, the theorem tells us that the integrability of the distribution defined above is equivalent to the eigenvectors $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$ of the map being smoothly extendible to a neighbourhood of p (which, in turn, amounts to the manifold having a genuine complex structure).

Discussion 2.3.1. *Let M be a complex manifold of dimension m. Then, T_pM has dimension $2m$ as a vector space over with basis $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^m}\}$. Its complexification, T_pM, is a $2m$-dimensional vector space over (note that this is no longer the tangent space of M); and has the following basis:*

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}\{\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\}, \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2}\{\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\}$$

*We can divide the space into a holomorphic and an anti-holomorphic half. The basis for T_p^*M is the obvious dual. (Note that we will get the same space if we complexify before or after dualizing, so the order of the * does not matter.)*
What we want to do is approach another notion of the exterior derivative. First, recall that on real manifolds,

$$dw = d(\sum a_{i_1,...,i_k} dx_{i_1} \wedge ... \wedge dx_{i_k}) = \sum \frac{\partial a_{i_1,...,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_k}$$

On a complex manifold, a differential form will be characterized by a holomorphic degree and an anti-holomorphic degree. So, an arbitrary $(r, s)-$ form will look like the following (in the standard basis):

$$\omega = \omega_{\mu_1...\mu_r\nu_1...\nu_s} dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}$$

Now,

$$d\omega = (\frac{\partial}{\partial z^\lambda}\omega_{\mu_1...\mu_r\nu_1...\nu_s} dz^\lambda + \frac{\partial}{\partial \bar{z}^\lambda}\omega_{\mu_1...\mu_r\nu_1...\nu_s} d\bar{z}^\lambda) \wedge dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}$$

It is evident that $d\omega$ is the sum of an $(r + 1, s)-$form and an $(r, s + 1)-$form. Accordingly, we write

$$d = \partial + \bar{\partial}$$

*The operators $\partial, \bar{\partial}$ are known as the **Dolbeault operators**.*

Definition 2.3.3 (Hermitian manifold)**.** A **Hermitian manifold** is a complex manifold M equipped with a Hermitian metric, i.e., a map on the complexified tangent bundle which assigns a Hermitian inner product on the fibres.

Definition 2.3.4 (Kähler form)**.** Let (M, g) be a Hermitian manifold. The two-form $\Omega := \frac{i}{2}(g - \bar{g})$ is called the **Kähler form**.

Definition 2.3.5 (Kähler manifold)**.** A **Kähler manifold** is a Hermitian manifold (M, g) whose Kähler form Ω is closed.

In a nutshell, a Kähler manifold is a manifold with three mutually compatible structures: A complex structure, a symplectic structure and a Riemannian structure.
There are, for example, two ways to apprach the construction of one:

- Complex manifold → Hermitian metric → Closed, compatible 2-form
- Riemannian manifold → Compatible complex structure → Closed, compatible 2-form

The compatibility between the metric and the symplectic form is via the definition of the Kähler form. It can also be read as $\Omega(X, Y) = g(JX, Y)$, when the metric starts off as real and we adorn it with a complex structure.
The compatibility between the metric and the complex structure is (in the second construction) given by the condition making it Hermitian, i.e., $g_p(J_pX_p, J_pY_p) = g_p(X_p, Y_p)$. In the first construction (which is what we used in our definitions), this is true by definition, and we only need to check it to be closed.

Lemma 2.3.2 (Kähler potential)**.** *Let (M, g, ω) be a Kähler manifold. Locally, there exists a real function K called the **Kähler potential** such that $\theta = -i\partial K$, where θ is the symplectic potential and ∂ is a Dolbeault operator.[14]*

Definition 2.3.6 (Fubini-Study metric)**.** On $\mathbb{CP}^n$, define the function $K := \log(1 + |\mathbf{z}|^2)$. This is a Kähler potential, and gives rise to the **Fubini-Study metric.**

The symplectic form associated with this is $\omega = i\partial\bar{\partial}K$, while the metric itself is explicitly given by $g_{i\bar{j}} := \frac{\partial^2}{\partial z_i \partial \bar{z}_j} K$.

3

Geometric quantization

This chapter is primarily based on [18], [8] and [16].

3.1 Prequantization

Let us restate our problem before proceeding.
In the classical framework, the state space is a symplectic manifold (M, ω) and observables are $C^\infty(M)$ functions. In the quantum framework, the state space is vectors in a Hilbert space $\mathscr{H}$ and the observables are Hermitian operators $\mathscr{O}$ on $\mathscr{H}$.
The question is how, given (M, ω), can one reconstruct $(\mathscr{H}, \mathscr{O})$, along with a map $C^\infty(M) \ni f \mapsto \hat{f} \in \mathscr{O}$ subject to the following conditions:

1. $f \to \hat{f}$ is linear
2. If f is constant, then $\hat{f}$ is the multiplication operator
3. If $\{f_1, f_2\} = f_3$, then $[\hat{f}_1, \hat{f}_2] = \hat{f}_3$.

These are Dirac's quantum conditions, put forward in [5]. There are some obvious Hilbert spaces and maps one can cook up (discussed in [18]), but they will fail to satisfy these. In fact, we will have to restrict ourselves to a particular class of symplectic manifolds, described by the following theorem.

Theorem 3.1.1 (Weil's integrality condition)**.** *Let (M, ω) be a symplectic manifold. Then, there exists a Hermitian line bundle $B \to M$ with a connection ∇ such that the curvature* *of ∇ is $\hbar^{-1}\omega \iff \varphi([\frac{\omega}{2\pi\hbar}]) \in H^2(M, \mathbb{Z}) \subseteq H^2(M, \mathbb{R})$, where φ is the isomorphism between the de Rham cohomology and the singular cohomology.*

Proof. We prove sufficiency first, followed by necessity.

- Sufficient: Suppose $\varphi([\frac{\omega}{2\pi\hbar}]) \in H^2(M, \mathbb{Z})$. Let its image in the Čech cohomology be $f = \{f_{jkl} = (\frac{1}{2\pi\hbar}(u_{jk} + u_{kl} + u_{lj}))\}$ (constructed per theorem 2.4). By assumption, the range of each f_{jkl} will be contained in $\mathbb{Z}$.
  Define $c_{jk} := e^{\frac{iu_{jk}}{\hbar}}$. It can be checked that these satisfy the cocycle conditions 2.3, and thereby determine a line bundle $B \to M$. Furthermore, since they satisfy $\frac{dc_{jk}}{c_{jk}} = \frac{i}{\hbar}(\theta_j - \theta_k)$, one can also reconstruct a connection ∇ with curvature $\hbar^{-1}\omega$ (see the discussion on connection potentials). The existence of a compatible Hermitian structure is a known result.

- Necessary: Suppose we have a Hermitian line bundle $B \to M$ and connection ∇ with curvature $\hbar^{-1}\omega$. It suffices to show (by the de Rham—Čech isomorphism) that $u_{jkl} = u_{jk} + u_{kl} + u_{lj}$ are integer-valued for $du_{jk} = \theta_j - \theta_k$, where $\frac{d\theta_j}{2\pi\hbar} = \omega$. (Note that the functions u_{jkl} are locally constant, which is why the Čech cohomology is well-defined.)
  But now note that $\theta_j - \theta_k = 2\pi i\hbar d(\log g_{jk})$ for transition functions g_{jk}. It follows that $u_{jkl} = \frac{1}{2\pi i\hbar}(\log g_{jk} + \log g_{kl} + \log g_{lj}) \implies e^{2\pi i\hbar u_{jkl}} = 1$ (using the cocycle condition) $\implies u_{jkl} \in \mathbb{Z}$, which completes the proof.

□

Definition 3.1.1. A symplectic manifold (M, ω) is said to be **quantizable** whenever ω satisfies the integrality condition.
The Hermitian line bundle $B \to M$ with connection ∇ is said to be the **prequantum bundle**.

Definition 3.1.2 (Kostant-Souriau prequantum operator)**.** Let (M, ω) be a quantizable symplectic manifold and $B \to M$ be the prequantum bundle.
Let $\mathscr{H}$ be the Hilbert space of square integrable sections $s : M \to B$ with inner product $\langle s, s' \rangle := \int_M (s, s')dV$ (where (s, s') is the inner product on the Hermitian line bundle).
Then, the operator $f \mapsto (\hat{f} : s \mapsto -i\hbar\nabla_{X_f} s + fs)$ is called the **prequantum operator**.

$\hat{f}$, as defined above, is Hermitian, and satisfies Dirac's quantum conditions.

Example 3.1.1 (Cotangent bundle)**.** *Let us consider the simplest classical observables, position and momentum, and see what the prequantized operators look like.*
Let $M = T^*Q$ for some configuration space Q having coordinates q^a. Give it an exact symplectic form $\omega = dp_a \wedge dq^a$ with symplectic potential $\theta = p_a dq^a$. Set $B = M \times \mathbb{C}, \nabla = d - i\hbar^{-1}\theta$ (with the obvious trivialization). It is clear that $\Omega = \hbar^{-1}\omega$.

1. $f = p_a : X_f = \frac{\partial f}{\partial p_a}\frac{\partial}{\partial q^a} - \frac{\partial f}{\partial q^a}\frac{\partial}{\partial p_a} = \frac{\partial}{\partial q^a}$.
   $-i\hbar\nabla_{X_f} + f = (-i\hbar)(\frac{\partial}{\partial q^a} - i\hbar^{-1}(p_a dq^a)(\frac{\partial}{\partial q^a})) + p_a = -i\hbar\frac{\partial}{\partial q^a}$.

2. $f = q^a$: *A similar computation as above yields* $\hat{q}^a = i\hbar\frac{\partial}{\partial p_a} + q^a$.

Example 3.1.2 (Torus)**.** *For a more involved example, let us consider the prequantization of the two-dimensional torus, as done in [9].*
Setup:

- $T^2 = \mathbb{R}^2/\mathbb{Z}^2$
- $\omega = N dx \wedge dy$
- $C^\infty(T^2) \cong$ *Doubly periodic functions on the plane, i.e.,* $f(x+m, y+n) = f(x,y), m, n \in \mathbb{Z}$

Now, the information we need to specify a prequantum bundle is an open cover on T^2 *with a collection of functions, along with a collection of one-forms. The functions have to satisfy the cocycle relations (this gives us a line bundle), and the one-forms have to be connection potentials (this gives us a connection—of course, which must also have the right curvature). For the open cover, consider* $U_\pm = \{(x,y) \in (\pm\delta, 1 \pm \delta) \times [0,1])\}$*. Identifying* $y = 0, y = 1$ *and* $x, x+1, x \in (-\delta, \delta)$ *returns the torus.*
The following are genuine transition functions (it can be checked that they satisfy the cocycle conditions):

$$c(x,y) = \left\{ \begin{array}{ll} 1, & x \in (\delta, 1-\delta) \\ e^{-i\frac{N}{\hbar}y}, & x \in (-\delta, \delta) \end{array} \right\}$$

With this, the following connection potentials will do the job:

$$\theta_\pm = \frac{N}{\hbar} x dy$$

It is immediate that the integrality condition is satisfied. A quick computation will establish that, for $\nabla = d - i\theta$*, the prequantum operator has the form*

$$(-i\hbar)(\frac{\partial f}{\partial x}[\frac{\partial}{\partial y} - \frac{N}{\hbar}x] - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}) + f$$

A final remark: The prequantum Hilbert space consists of square-integrable maps satisfying $\phi(x+m, y+n) = e^{i\frac{N}{\hbar}}\phi(x,y)$ *(this follows from the form of the transition functions).*

The trouble with prequantization is that, in general, the Hilbert space obtained is too 'big'. For example, say we want to take a free particle in phase space to a quantum-mechanical free particle. Our ambient manifold is $\mathbb{R}^{2n}$, and the quantum-mechanical system is an irreducible representation of the Heisenberg algebra. However, a prequantization of $\mathbb{R}^{2n}$

yields only a *reducible* representation of the Heisenberg algebra; the quantized position and momentum operators have invariant subspaces.
Similarly, a prequantization of S^2 comes nowhere close to yielding an irreducible representation of $u(2)$ (for the theory of spin).
One way to deal with this is by using polarizations.

3.2 Polarization

Definition 3.2.1 (Complex polarization)**.** Let (M, ω) be a symplectic manifold. A **complex polarization** P on M is a distribution on the complexified tangent bundle $T_{\mathbb{C}}M$ such that:

- Each $P_m \subseteq (T_m M)_{\mathbb{C}}$ is a Lagrangian subspace
- The dimension of $D = P \cap \bar{P} \cap TM$ is constant
- P is integrable

Discussion 3.2.1. *Any Kähler manifold admits two natural polarizations.*

- ***Holomorphic polarization:*** *Let* $P = \{X \in T_{\mathbb{C}}M | JX = iX\}$. *It can be checked that this is a Kähler polarization; and is, in fact, generated by the holomorphic half of the tangent space basis.*
- ***Anti-holomorphic polarization:*** *We have* $\bar{P} = \{X \in T_{\mathbb{C}}M | JX = -iX\}$. *It can be checked that this is a Kähler polarization; and is, in fact, generated by the anti-holomorphic half of the tangent space basis.*

Definition 3.2.2 (Polarized section)**.** Let P be a complex polarization on a symplectic manifold M with vector bundle B. A smooth section $s : M \to B$ is said to be polarized if $\nabla_{\bar{X}} s = 0$ for every $X \in V_P(M)$.

Going back to our problem: We can choose a polarization P and try cutting down our Hilbert space of prequantization down to only those square-integrable smooth sections which are polarized with respect to P.
However, there are two issues in the general situation: Existence and closure. A quick example will help illustrate the first.

Example 3.2.1. *Let* $M = T^*Q$ *for any* $n-$*dimensional smooth manifold* Q*, and equip it with the canonical coordinates* $\{q_1, ..., q_n, p_1, ..., p_n\}$ *and the usual projection map* $\pi : M \to Q$.
Let $D = \ker(d\pi) = \text{span}\{\frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_n}\}$.

- *Leaves: A basic theorem tells us that* $\ker(d\pi_p) = T_p\Lambda_p$*, where* $\Lambda = \pi^{-1}(p)$*. Thus, our leaves are the submanifolds given by* $\Lambda_x = \pi^{-1}(x)$ *for* $x \in Q$*; in other words, they are the fibres* T_pQ *of the projection.*

- *Space of leaves: If we identify each of the fibres in the cotangent bundle to a point, it pinches down to our original manifold. Therefore,* $M/D \sim Q$*.*

- *Polarized sections: Our polarized sections are ones of the form* $\frac{\partial s}{\partial p_i} = 0 \implies s \equiv s(q)$*.*

For the sake of concreteness, let $Q = S^1$*. Then, each leaf* $\Lambda_p \equiv T_pS^1$*. Are polarized sections square integrable here?*

$$\int_M \langle s, s\rangle dV = \int_{T^*S^1} \langle s(q), s(q)\rangle dp_i \wedge dq_i = \int_{S^1} \langle s(q), s(q)\rangle dq_i \int_{TS^1} K dp_i = \infty$$

because our sections will be of some constant value K *over* TS^1*—which, in turn, is not compact.*

So, even in this relatively simple example, there turn out to be no nonzero square-integrable polarized sections. One can try to remedy this by restricting integration over only the relevant variables; formally, this procedure is known as *metaplectic correction*. We will not discuss it in more detail here.
Moreover, a quantized operator $\hat{f}$ will not, in general, map polarized sections to polarized sections. One can compute that:

$$\nabla_{\bar{X}} \hat{f} = [\nabla_{\bar{X}}, \hat{f}] + \hat{f}\nabla_{\bar{X}} = \hat{f}\nabla_{\bar{X}} - i\hbar\nabla_{[\bar{X}, X_f]}$$

Therefore, something one might want to ensure is that the Hamiltonian vector field X_f of the observable f 'preserves' $\bar{P}$ (as this can be seen to ensure $\hat{f}s$ being polarized).

3.3 Obstructions

There is an alternative way of framing the problems of irreducibility we saw above. This line of investigation was initiated in [8],[9] and other related works.

Definition 3.3.1 (Basic set)**.** Let (M, ω) be a symplectic manifold. A **basic set** is a finite-dimensional linear subspace $\mathscr{B} \subset C^\infty(M)$ such that:

1. $1 \in \mathscr{B}$

2. $\mathscr{B}$ is transitive & minimal

Definition 3.3.2 (Quantization)**.** Let (M, ω) be a symplectic manifold, and $\mathscr{B} \subset \mathscr{O} \subset \mathscr{P} = C^\infty(M)$ be such that $\mathscr{O}$ is a Poisson subalgebra and $\mathscr{B}$ is a basic set. Then, a **quantization** of the pair $(\mathscr{O}, \mathscr{B})$ is an operator Q such that:

- $Q : \mathscr{O} \to \text{Herm}(\mathscr{H})$ is a prequantum operator
- $Q(\mathscr{B})$ is irreducible & integrable

where, in the first condition, the right-hand side is Hermitian operators on $\mathscr{H}$, the Hilbert space of square integrable sections on a prequantum bundle $B \to M$.

Let us clarify some of the notions mentioned above.

1. Transitive: The set $\{X_f | f \in \mathscr{B}\}$ spans the tangent spaces to M everywhere. The idea is to have a 'complete' set of classical observables.

2. Minimal: There exists no transitive subspace of $\mathscr{B}$.

3. Irreducible: The idea is for the quantized observables to have no invariant subspaces in $\text{Herm}(\mathscr{H})$. An alternative recasting of this—using Schur's lemma—would be to require that scalar matrices are the only elements of $\text{Herm}(\mathscr{H})$ which commute with the quantized observables.

4. Integrable: This condition essentially requires that our Lie algebra representation be extendible to a Lie *group* representation in the following sense: Given a Lie algebra representation $Q : \mathscr{P}(\mathscr{B}) \to \text{Herm}(\mathscr{H})$ (where $\mathscr{P}(\mathscr{B})$ is the Poisson algebra generated by $\mathscr{B}$), we require that there exist a Lie group with unitary representation $\Pi : G \to \text{Herm}(\mathscr{H})$ such that $T_eG \cong \mathscr{P}(\mathscr{B})$ and $d\Pi = Q$.

A *full quantization* is one in which $\mathscr{O} = C^\infty(M)$. It was proven in [8] that there are no nontrivial full quantizations of $(C^\infty(\mathbb{R}^{2n}), h(2n))$ or $(C^\infty(S^2), u(2))$. In a sense, this is nothing but a restatement of the remarks made at the end of section 3.1. We will sketch a proof of the latter result here.

Lemma 3.3.1. *There is no non-trivial full quantization of* $(C^\infty(S^2), u(2))$.

Proof. Let Q be a quantization of the above on a prequantum bundle $B \to M$ with Hilbert space $\mathscr{H}$, and $\{1, S_1, S_2, S_3\}$ be the standard basis set for $u(2)$. The symplectic form is:

$$\omega = \frac{1}{2s^2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} S_i dS_j \wedge dS_k$$

where $s^2 = S_1^2 + S_2^2 + S_3^2$. From this, it is immediate that the Poisson bracket takes the following form:

$$\{f, g\} = -\sum_{i,j,k=1}^{3} \epsilon_{ijk} S_i \frac{\partial f}{\partial S_j} \frac{\partial g}{\partial S_k}$$

(Note that we may understand S_i as a $C^\infty(M)$ function via the fact that the manifold can be realized as a coadjoint orbit of this Lie group. The symplectic form is likely nothing but the one associated with the Fubini-Study metric.)

From the above, we have that, in particular:

$$\{S_j, S_k\} = -\sum_{l=1}^{3} \epsilon_{jkl} S_l$$

Then, by assumption:

$$[Q(S_j), Q(S_k)] = i\hbar \sum_{l=0}^{3} \epsilon_{jkl} Q(S_l) \tag{3.1}$$

Also,

$$Q(\sum_{i=1}^{3} S_i^2) = s^2 I \tag{3.2}$$

Finally,

$$\sum_{i=1}^{3} Q(S_i)^2 = \hbar^2 j(j+1) I \tag{3.3}$$

We claim that (3.1), (3.2), (3.3) together produce a contradiction. The last one utilized integrability and irreducibility in its derivation. It can be shown that they, together with irreducibility, lead to the following equation:

$$Q(S_i S_k) = \frac{a}{2}(Q(S_i)Q(S_k) + Q(S_k)Q(S_i)) \tag{3.4}$$

If we follow these rules and quantize the following two equations, we will arrive at a contradiction:

$$s^2 S_3 = \{S_1^2 - S_2^2, S_1 S_2\} - \{S_2 S_3, S_3 S_1\}$$

$$2s^2 S_2 S_3 = \{S_2^2, \{S_1 S_2, S_1 S_3\}\} - \frac{3}{4}\{S_1^2, \{S_1^2, S_2 S_3\}\}$$

This completes the proof. □

The derivation of equation 3.4 is highly lengthy and technical, but can be found in [10]. At any rate, we are now in a position to ask some interesting questions.

Question: What do these obstruction results look like in the language of polarizations?

Conjecture: Let $B \to S^2$ be a prequantum bundle and $\mathscr{H} = L^2(C_B^\infty(S^2))$. Then, there is no polarization $\mathscr{H}_P$ of $\mathscr{H}$ such $\sigma : u(2) \to \mathrm{Herm}(\mathscr{H}_P)$ is an irreducible representation.

Question: Noting that $\mathbb{CP}^1 = S^2$, is there an obstruction to a full quantization of $\mathbb{CP}^n$ for all $n \in \mathbb{N}$?

Conjecture: There exists no full quantization of $\mathbb{CP}^n$ for $\mathscr{B} = u(n+1)$.

Finally, one can also try to generally classify spaces for which a non-trivial full quantization exists. In [8], it is conjectured that if we choose a basic set $\mathscr{B}$ such that $P(\mathscr{B})$, the Poisson algebra generated by $\mathscr{B}$, is dense in $C^\infty(M)$, then there exists a non-trivial full quantization.

3.4 Quantization of states

We have completed discussing an approach to quantizing observables. Now, let us go further and try to quantize *states* as well. A classical state can be understood merely as a point on the manifold; however, for a quantum-mechanical state, the appropriate framework cannot merely be our Hilbert space. This is because quantum-mechanical states are essentially the same upon being multiplied by a scalar. Therefore, our idea will be to construct an embedding $M \to \mathbb{CP}(\mathscr{H})$, the *projective* Hilbert space. The construction discussed here is based on [16].
For our setup, let M be a Kähler manifold with prequantum bundle $M \to B$. Let us also impose a polarization and denote by $\Gamma_F(E)$ the space of holomorphic global sections (where $F = T^{(0,1)}M$).

Definition 3.4.1 (Quantum states)**.** The complex Hilbert space of quantum states $\mathscr{H}$ consists of holomorphic sections $s : M \to E \otimes T^{*(n,0)}(M)$ such that $\langle s, s \rangle < \infty$, where $\langle s, t \rangle := i^{n^2} \int_M (s, t)$ and (s, t) is from the Hermitian structure on the line bundle.

In geometric quantization, the Hilbert space was usually simply defined as (a subspace of) square-integrable sections, where integration was against the natural top-form associated with a symplectic manifold. In this case, allowing our section to choose an $n-$form as well gives us some extra freedom in what volume form or measure to integrate against.

Definition 3.4.2 (Quantizable observables)**.** The Lie algebra of quantizable observables is defined as $C_{FF} := \{f \in C^\infty(M) : [X_f, F] \subseteq F\}$.

Recall from the discussion in the previous section that this is done in order to ensure that the quantized operator maps polarized sections to polarized sections.

Definition 3.4.3 (Kostant-Souriau quantization operator)**.** The quantization operator is given by $f \mapsto \hat{f} := \frac{1}{i}[(\nabla_{X_f} + if) \otimes \mathscr{L}_{X_f}]$

The first half of $\hat{f}$ acts on the 'pure' section, while the Lie derivative acts on the differential form.

Lemma 3.4.1. *$\hat{f}$ is a Hermitian operator.[16]*

We are now ready to define the embedding we were looking for. Firstly, note that in a local trivialization (U_α, τ_α), we may write $s = \psi_\alpha s_\alpha \otimes dz_\alpha^1 \wedge ... \wedge dz_\alpha^n, \psi_\alpha \in C^\infty(M)$.
For a fixed z in this local trivialization, consider the 'evaluation' functional $H_z : s \mapsto \psi_\alpha(z)$. This is an element of $\mathscr{H}^*$. If we show that this is a *continuous* linear functional, it follows from the Riesz representation theorem that

$$H_z(s) = \psi_\alpha(z) = \langle K_{\bar{\alpha}}(\bar{z}, \cdot), s \rangle$$

for some $K_{\bar{\alpha}} \in \mathscr{H}$. Since H_z depends on $z, K_{\bar{\alpha}}$ will, in general, depend on $\bar{z}$ (since the inner product is Hermitian). This is what gives us our embedding:

$$M \to \mathbb{CP}(\mathscr{H}), z \mapsto [K_{\bar{\alpha}}(\bar{z}, \cdot)]$$

Lemma 3.4.2. *H_z is continuous.*

Proof. We prove that $|H_z(s)|= |\varphi_\alpha(z)|\leq c_\alpha||s||$ for some positive constant c_α. Then, H_z will be continuous by virtue of being bounded.
$||s||= (\int_M (s, s))^{\frac{1}{2}} = (\int_{U_\alpha} |\varphi_\alpha|^2||s_\alpha||^2 dz_\alpha^1 \wedge ... \wedge dz_\alpha^n \wedge d\bar{z}_\alpha^1 \wedge ... \wedge d\bar{z}_\alpha^n)^{\frac{1}{2}} \geq$
$\int_{U_\alpha} |\varphi_\alpha|\cdot||s_\alpha|| dz_\alpha^1 \wedge ... \wedge dz_\alpha^n \wedge d\bar{z}_\alpha^1 \wedge ... \wedge d\bar{z}_\alpha^n = ||\varphi_\alpha||_1 \cdot ||s_\alpha||_1$, where we used Hölder's inequality in the middle.
We know that $0 < ||s_\alpha||_1 < \infty$. Therefore, taking it to the right-hand side, we get $c_\alpha||s||\geq ||\varphi_\alpha||_1$.
But finally, recall that we had assumed φ_α is holomorphic. Then, for any $z \in D \subseteq U_\alpha, |\varphi_\alpha(z)| \sim \int_D \varphi_\alpha$ by Cauchy's theorem. This completes the proof. □

Definition 3.4.4 (Reproducing kernel)**.** The reproducing kernel function $K_{\bar{\alpha}\beta}(\bar{z}, \cdot)$ is defined as the $C^\infty(M)$ function attached locally to $K_{\bar{\alpha}}(\bar{z}, \cdot)$ viewed as a section:

$$K_{\bar{\alpha}}(\bar{z}, v) = K_{\bar{\alpha}\beta}(\bar{z}, v) s_\beta \otimes dv_\beta^1 \wedge ... \wedge dv_\beta^n$$

What we wanted to do was not just map, but *embed* classical phase space into quantum phase space. For this, we need to characterize when the differential of the map we have defined will be nonsingular. This turns out to be true if the map is injective and a certain special quadratic form is positive-definite.
Another remark deserves being made here: We would want two distinct points $z_1, z_2 \in M$ to map to distinct elements in the projective Hilbert space. The following condition would ensure this: For all $z_1, z_2 \in M$, there are sections s_1, s_2 such that $[\psi_{1\alpha}(z_1)\psi_{2\beta}(z_2) - \psi_{1\beta}(z_2)\psi_{2\alpha}(z_1)] \neq 0$. (Otherwise, the relevant sections would have been multiples of each other and thus equal in projective space.)

4

Deformation quantization

This chapter is primarily based on [4], [7] and [3].

4.1 Idea

As initiated in [2], the idea behind deformation quantization is to treat quantization as *"a deformation of the structure of the algebra of classical observables, rather than as a radical change in the nature of the observables."*

Abstractly, its aim is to equip the classical algebra of observables with a structure (a *formal deformation*, most commonly referred to as the $*$—product) which produces an associative algebra on $C^\infty(M)$ over some formal parameter (most commonly, $\hbar$).

This associative algebra, with its commutator, is supposed to formulate quantum theory; and, under a certain limit, is expected to reproduct the Poisson bracket *as well as* the pointwise product on the original classical algebra.

One can see some immediate advantages to this approach.

- Unlike geometric quantization, deformation quantization is not indifferent to the pointwise multiplicative structure on $C^\infty(M)$.
- While geometric quantization has various obstructions, deformation quantization is possible on any symplectic manifold.

While deformation quantization is to be understood on its own terms, we shall also (following [3]) define a particular $*$—product on $\mathbb{CP}^n$ and describe its construction

explicitly *in the backdrop of* geometric quantization.
A natural way of approaching the study of deformation quantization is by studying deformations on the following increasingly general structures:

$$(\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^d, \pi) \to (M, \omega) \to (M, \pi)$$

where ω is a symplectic structure and π is a Poisson structure.
The first of these is the Moyal product (on the Weyl algebra); the movement from this to any symplectic manifold (M, ω) is what was done in [6]. The fact that a deformation quantization exists for any arbitrary Poisson manifold (M, π)—the final rung on the ladder—was proven by Maxim Kontsevich in 1997 [11], and is (in some sense) modelled on Fedosov's first generalization.

Definition 4.1.1 (Formal deformation)**.** Let A be a commutative associative algebra with unit over some commutative base ring R, and $\hbar$ be a formal parameter. Then, a **formal deformation** of A is the algebra $A[\hbar]$ of formal power series over the ring $k[\hbar]$ of formal power series.

Elements of the deformed algebra are of the form $\sum c_i\hbar^i, c_i \in A$. Their product is given by $(\sum a_i\hbar^i) \cdot_\hbar (\sum b_j\hbar^j) = \sum(a_{r-l}b_l)\hbar^r$.

Definition 4.1.2 (Star product)**.** A **star product** is a $k[\hbar]$-linear associative product $\star$ on $A[\hbar]$ which deforms the trivial extension; that is, such $v \star w = v \cdot_\hbar w$ modulo $\hbar$ for $v, w \in A[\hbar]$.

In the case of Poisson manifolds, we are concerned with formal deformations and star products with a particular structure.

Definition 4.1.3. A star product on a Poisson manifold (P, π) is an $\mathbb{R}[\hbar]-$bilinear map $\star : C^\infty(P)[\hbar] \times C^\infty(P)[\hbar] \to C^\infty(P)[\hbar]$ such that:

1. $f \star g = fg + \sum_{i \geq 1} B_i(f, g)\hbar^i$
2. $(f \star g) \star h = f \star (g \star h)$
3. $1 \star f = f \star 1$

where the B_i are (bounded) bidifferential operators on $C^\infty(P)$.

Example 4.1.1 (Moyal product)**.** *Consider the standard symplectic manifold* $(\mathbb{R}^{2n}, \omega_0)$ *with local Darboux coordinates. The Moyal product is defined as:*

$$(f \star g)(x) = \exp(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial q^i}\frac{\partial}{\partial p^j})f(x)g(y)|_{y=x}$$

(where note that, since we are in the Darboux coordinates, the components ω^{ij} *will be constants). It is straghtforward to check that this satisfies the three conditions above (using the Baker-Campbell-Hausdorff formula to verify the first).*

4.2 Berezin quantization of $\mathbb{CP}^n$

Recall the construction of the hyperplane bundle $H \to \mathbb{CP}^n$ in example 2.2.2.

Lemma 4.2.1. *The curvature form on $H^{\otimes m}$ is given by $m\Omega_{FS}$, where Ω_{FS} is the Kähler form associated with the Fubini-Study metric and $H^{\otimes m}$ is the m-th tensor power of H.[4]*

Discussion 4.2.1. *Let $[1, \mu_1, ..., \mu_n]$ be homogeneous coordinates for U_0, and let*

$$\Psi_{(q_1,...,q_n;q)}(\mu) = \frac{1}{\sqrt{D_{(q_1,...,q_n;q)}}}\mu_1^{q_1}...\mu_n^{q_n}$$

where $\sum q_i = q, q \in \{0, 1, ..., m\}$ and

$$D_{(q_1,...,q_n;q)} = c(m)\int_{U_0} \frac{|\nu_1|^{2q_1}...|\nu_n|^{2q_n}}{(1+|\nu|^2)^m} dV(\nu)$$

where

$$(c(m))^{-1}\int_{U_0} \frac{1}{(1+|\nu|^2)^m} dV(\nu)$$

where $\Phi_{FS}(\mu, \bar{\nu}) = \ln(1 + \mu \cdot \bar{\nu})$ and

$$dV(\mu) = |\Omega_{FS}^n(\mu)|_{U_0}| = \frac{|d\mu \wedge d\bar{\mu}|}{(1+|\mu|^2)^{n+1}}$$

using the fact that $\mathbb{C}^n \equiv U_0 \subset \mathbb{CP}^n$. Finally, define the following inner product on the function space of U_0:

$$\langle f, g\rangle = c(m)\int_{U_0} \frac{\overline{f(\nu)}g(\nu)}{(1+|\nu|^2)^m} dV(\nu)$$

Lemma 4.2.2. *$\{\Psi_{(q_1,...,q_n;q)}\}$ forms an orthonormal basis for sections of $H^{\otimes m}$ on $U_0 \subset \mathbb{CP}^n$.[4]*

We have described sections on $U_0 \subset \mathbb{CP}^n$. The result is essentially the same for the remaining $\{U_i\}$, which together cover $\mathbb{CP}^n$.

Discussion 4.2.2 (Coherent states)**.** *There are two technically distinct but ultimately identical ways to approach the definition of coherent states. In the setup, let (M, ω) be a quantizable symplectic manifold with polarization P, L a Hermitian line bundle, and $\mathscr{H}$ the set of square integrable polarized sections.*

- ***General case:** For any $q \in L_0$, define $l_q \in \mathscr{H}^*, l_q(s)q = s(\pi(q))$. That it is a linear continuous functional follows from considerations similar to lemma 3.4.2. We can therefore once again apply the Riesz representation theorem to get the following sections e_q out:*

$$\langle s, e_q\rangle = l_q(s)$$

These e_q are known as the coherent states. Taking their inner product with a given section s evidently acts as a kind of evaluation on s.

We have, on the other hand,

- $\mathbb{CP}^n$*: More explicitly, coherent states are defined here as*

$$\psi_\mu(\nu) := \sum_{\sum q_i = q, q\in\{0,\dots,m\}} \overline{\Psi_{(q_1,\dots,q_n;q)}(\mu)}\Psi_{(q_1,\dots,q_n;q)}(\nu)$$

It is easy to prove (via orthonormality) that they satisfy the following so-called 'reproducing kernel property': $\langle\psi_\mu, \Psi\rangle = \Psi(\mu)$.

The connection is not too difficult to see:

$$[\langle\psi_\mu, \Psi\rangle = \Psi(\mu)] \equiv [\langle e_q, s\rangle = l_q(s)]$$

Definition 4.2.1 (Symbol)**.** Let $\hat{A} : \mathscr{H} \to \mathscr{H}$ be a bounded operator. Then, we define its **symbol**, A, to be the following $C^\infty(M)$ function:

$$A := \frac{\langle\hat{A}\psi_\mu, \psi_\mu\rangle}{||\psi_\mu||^2}$$

In the other notation, this is $\frac{\langle\hat{A}e_q, e_q\rangle}{||e_q||^2}$.
Having established this much, the noncommutative star product is finally within our reach. It is defined in the following manner:

$$A_1 \star A_2 := A_1 \circ A_2,$$

that is, it is the symbol of the operator $\hat{A}_1 \circ \hat{A}_2$.
The fact that this is indeed a deformation of the $C^\infty(M)$ algebra under an appropriate limit is what is established by the next theorem.

Theorem 4.2.3 (Berezin)**.** *The following limits hold almost everywhere:*

1. $\lim_{m\to\infty}(A_1 \star A_2)(\mu) = A_1(\mu)A_2(\mu)$
2. $\lim_{m\to\infty} m(A_1 \star A_2 - A_2 \star A_1)(\mu) = i\{A_1, A_2\}(\mu)$*[3]*

In general, the Berezin symbol gives us a way to go downstairs—to associate a classical observable, a $C^\infty(M)$ function, with a Hermitian operator acting on the Hilbert space of prequantization. On the other hand, the Kostant-Souriau prequantum operator went upstairs, taking a classical observable and giving out a Hermitian operator. There is a natural question to ask about how these two processes interact.

Theorem 4.2.4. *Let (M, ω) be a quantizable symplectic manifold with polarization F and Hermitian line bundle with connection (L, h, ∇). Then, given a quantizable function $f \in C^\infty(M)$, the following schema holds:*

$$f \xrightarrow[Kostant-Souriau]{} (-i\hbar\nabla_{X_f} + f) \xrightarrow[Berezin]{} (-i\hbar X'_f(\ln\theta) + f)$$

where X'_f is the $F-$component of X_f.[17]

The function $\theta \in C^\infty(M)$ is defined as $\theta(x) = |q|^2||e_q||^2, q \in L_0, \pi(q) = x$.
It is easy to see that this is well-defined: For let $\pi(q) = \pi(q') = x$, and $q' = cq$ (since we're dealing with a line bundle). On the other hand,
$l_{cq}(s) \cdot cq = s(x) = l_q(s) \cdot q \implies l_{cq}(s) = c^{-1}l_q(s) \implies e_{cq} = c^{-1}e_q$; and so the product remains unchanged.
In particular, then, $\theta(x) = |s_0(x)|^2||e_{s_0(x)}||^2$, from which continuity follows.

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